

## 6.6 Equilibrium Values and their Stability for First-Order Non-Linear DDS

In Section 6.5 we have derived a formula to determine the equilibrium value for a first-order linear DDS and also a criterion to determine the stability of the equilibrium value. Unfortunately, we will not be able to derive such a general formula for a first-order non-linear DDS. However, we can use the same approach for finding the equilibrium as we did in case of a linear DDS. Recall that for a system in equilibrium, the new output value is equal to the old output value, i.e., no change occurs. Thus, if the system is given in the form

$$x(n+1) = f(x(n))$$

then this requirement for the equilibrium becomes

$$x(n) = x(n+1) = f(x(n)).$$

If we again denote the equilibrium value by  $x$ , we need to solve the following equation:

$$x = f(x).$$

A value that solves this equation is called a *fixed point* of the function  $f$ . We summarize this in the following theorem.

### Theorem 5 (Equilibrium Value for Non-Linear First-Order DDS)

The equilibrium values of a general first order DDS of the form

$$x(n+1) = f(x(n))$$

are the fixed points of the function  $f$ . The equilibrium values can be computed by solving the equation

$$x = f(x)$$

for  $x$ .

Here is an example:

We want to find the equilibrium value(s) for the DDS given by  $x(n+1) = 5x(n)^2$ . The model function  $f$  (= right hand side of the iterative model equation) is given by  $f(x(n)) = 5x(n)^2$ . To find the equilibrium, we have to solve the following equation:

$$x = f(x) \quad \text{or} \quad x = 5x^2.$$

To find the solution(s), we can use either the palette function **SolveIt** or analytical methods (depending on your mathematics background). Here is how to do this with **SolveIt**. Recall that

**SolveIt** has as entries the left hand side and the right hand side of the equation, as well as the variable to be solved for.

`SolveIt[x, 5x2, x]`

The resulting output is

`{{x → 0}, {x → 1/5}}`.

This indicates that both  $x = 0$  and  $x = \frac{1}{5}$  are equilibrium values. We can verify this by substituting these values into the original equation.

To solve the above equation analytically, we use methods to solve quadratic equations. We start by moving all terms to one side of the equation. The next step is to look whether factoring is possible. If not, apply the quadratic formula (see appendix, A4).

$$\begin{aligned} x &= 5x^2 \\ \Rightarrow 0 &= 5x^2 - x \\ \Rightarrow 0 &= x(5x - 1) \end{aligned}$$

If a product is zero, then either one or both of the factors has to be zero. This implies that either

$$\begin{aligned} x = 0 \quad \text{or} \quad 5x - 1 = 0 \\ \Rightarrow x = 0 \quad \text{or} \quad 5x = 1 \\ \Rightarrow x = 0 \quad \text{or} \quad x = \frac{1}{5} \end{aligned}$$

Again, both  $x = 0$  and  $x = \frac{1}{5}$  are the equilibrium values.

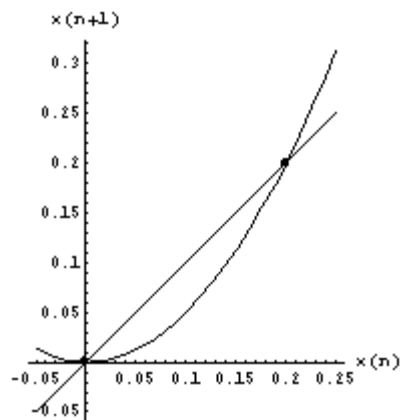
### Activity 6.6.1

For the DDS given below, find the equilibrium value(s) by using either the palette function **SolveIt** or appropriate analytical methods.

- a)  $x(n+1) = x(n) \cdot (1 - x(n))$
- b)  $x(n+1) = x(n) - 2x(n)^2 + 3x(n)^3$
- c)  $x(n+1) = x(n)^2 + x(n)^3$

Graphically, we can find fixed points of the function  $f$  by drawing the graph of the function together with the line  $y = x$ , the  $45^\circ$  line. The x-values of the points at which the graph intersects

the  $45^\circ$  line are the fixed points or equilibrium values. Why is that the case? Remember that in this graph, the input value is given by  $x(n)$  (on the horizontal axis), and the output value is given by  $f(x(n))$  (on the vertical axis). At the  $45^\circ$  line, the two values are identical, hence  $x(n) = f(x(n)) = x(n+1)$ . Thus, the system is in equilibrium for this input value. We will illustrate this method for the example above,  $x(n+1) = 5x(n)^2$ .



From the graph, we identify the points of intersection,  $(0,0)$  and  $(0.2, 0.2)$ , respectively. Their input values, namely 0 and  $0.2 = 1/5$ , are the two equilibrium values.

We now turn to determining the stability of the equilibrium values for a non-linear DDS. The criterion is still the same:

- If the initial value is slightly above or below the equilibrium value and the sequence of system values converges (= gets close) to that equilibrium value, then the equilibrium is **stable**.
- If the initial value is slightly above or below the equilibrium value and the sequence of system values diverges away from that equilibrium value, then the equilibrium is **unstable**.
- If the sequences of system values for initial value slightly above or below the equilibrium value exhibit both converging and diverging behavior, then the equilibrium is **semi-stable**. In all other cases, the equilibrium is **neutral**.

To determine stability, we can look at a table of values or a Cobb-web diagram. Let's start with the table. In the previous example,  $x(n+1) = 5x(n)^2$ , we found that the equilibrium values are  $x = 0$  and  $x = 0.2$ . We first check the equilibrium value  $x = 0$  for stability. We can create a sequence of values using the palette function **IteratedValueSeq**. As starting values we can use 0.1 and -0.1. (Be careful not to choose a value that is bigger than the second equilibrium value 0.2.) In order to graph the sequences later, we give each one a name and also create a sequence starting at the equilibrium value.

```

A1= IteratedValueSeq[5x^2, x, 0.1, 10, 1]
A2 = IteratedValueSeq[5x^2, x, -0.1, 10, 1]
A3 = IteratedValueSeq[5x^2, x, 0, 10, 1]

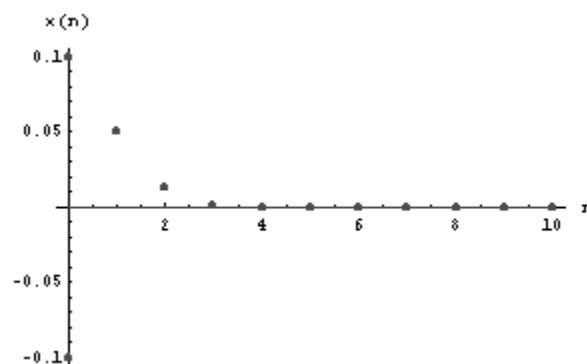
```

The table below summarizes the results (to six decimal places).

$n$	A1 $x(n)$	A2 $x(n)$	A3 $x(n)$
0	<b>0.1</b>	<b>-0.1</b>	<b>0</b>
1	0.05	0.05	0
2	0.0125	0.0125	0
3	0.000781	0.000781	0
4	0.000003	0.000003	0
5	0.000000	0.000000	0
6	0.000000	0.000000	0
7	0.000000	0.000000	0
8	0.000000	0.000000	0
9	0.000000	0.000000	0
10	0.000000	0.000000	0

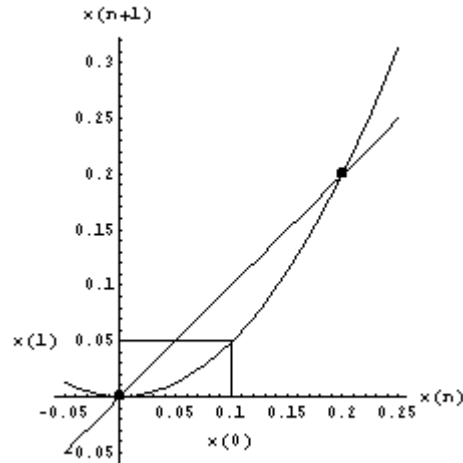
Since the values for  $x(n)$  approach zero, no matter whether we start slightly above or slightly below the equilibrium value,  $x = 0$  is a stable equilibrium. We can visualize this behavior by using the built-in *Mathematica* function **Show**. Note that the values for the equilibrium sequence are connected by a line because we used the (optional) value of **True** in the last **ListGraph** command.

```
Show[ListGraph[A1], ListGraph[A2], ListGraph[A3, True]]
```

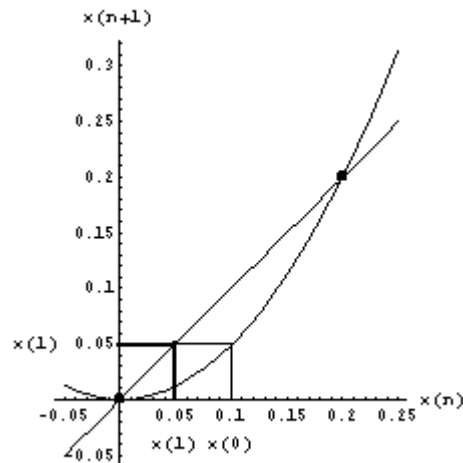


Now we will look at how a *Cobb-web diagram* or (*Cobb-web graph*) is constructed. We start by graphing the model function  $f$  together with the  $45^\circ$  line (where  $x(n+1) = x(n)$ ). The horizontal axis displays the values for  $x(n)$  (=input values) and the vertical axis displays the values for  $x(n+1)$  (= output values). We will use horizontal and vertical lines to read off the next output

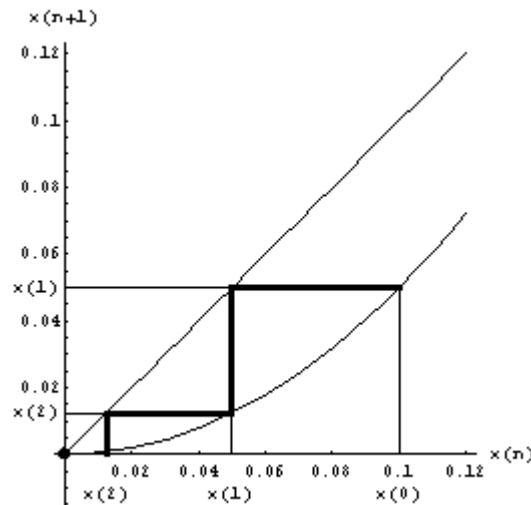
value and to translate this value from the vertical axis to the horizontal axis (so it becomes the next input value). First, we draw a vertical line through the initial value  $x(0)$  until this line intersects with the graph of the model function  $f$ . The corresponding output value can be read off by drawing a horizontal line from the intersection point to the vertical axis. This gives the value of  $x(1)$ . Below is an illustration of the procedure for the iterative model equation  $x(n+1) = 5x(n)^2$  with initial value  $x(0) = 0.1$ . In this case,  $x(1) = 0.05$ .



In order to use  $x(1)$  as the next input value, we have to transfer its value from the vertical to the horizontal axis. We could just read it off, and then mark the appropriate value on the horizontal axis. Alternatively, we can use the  $45^\circ$  line (where input and output values are identical) to assist with the transfer. To make the output value into an input value, we draw a horizontal line from  $x(1)$  to the  $45^\circ$  line, then a vertical line down to the horizontal axis (thick lines in the graph below). This transforms the value of  $x(1)$  from output to input value.



We repeat this process, now starting with  $x(1)$ , until we have transferred the value of  $x(2)$  to the horizontal axis. You may notice that each step creates a number of lines, parts of which are drawn twice. Those segments, as well as the initial vertical line, are shown as thin lines in the next graph. All other segments are drawn as thick lines; they constitute the Cobb-web diagram and are the only ones usually drawn.



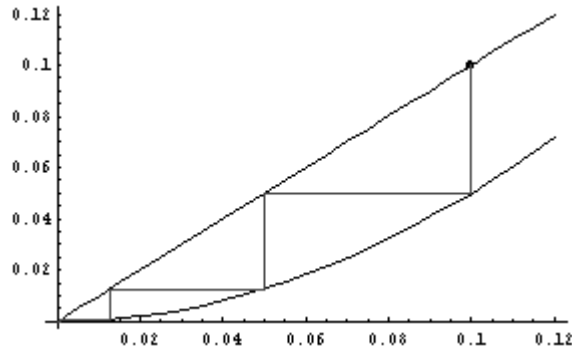
Such a Cobb-web diagram can be created with the palette function **MapIt**. The function **MapIt** needs as its entries the following:

#### ?MapIt

The function `MapIt[{func, x}, xinit, n, {xmin, xmax}, focus, percent]` displays  $n$  iterations of the function `func` expressed in terms of `x`. The iteration starts at  $x = xinit$  and is displayed for  $x$ -values from `xmin` to `xmax`. When zooming in, the graph is centered at  $x = focus$ . The value of `percent` determines the amount of zooming: For  $0 < percent < 1$ , we zoom in, for  $percent > 1$ , we zoom out.

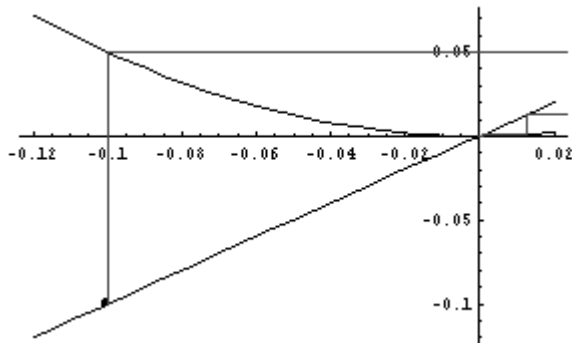
Let's look at the different entries one by one for our example. The model function is  $f(x) = 5x^2$ , thus **func** =  $5x^2$ , and the variable used is **x**. The initial value is **xinit** = **0.1**. To see 5 iterations, we set **n** = **5**. The table of values (= A1) and the previous graph indicate that choosing **{0, 0.12}** as the range for the horizontal axis will show all intermediate steps. We can choose the midpoint on the horizontal axis as the center point, i.e., **focus** =  $(0 + 0.12)/2 = 0.06$ , and no zooming, i.e., **percent** = **1**. This leads to the following use of **MapIt**:

```
MapIt[{5x^2, x}, 0.1, 5, {0,0.12}, 0.06, 1]
```

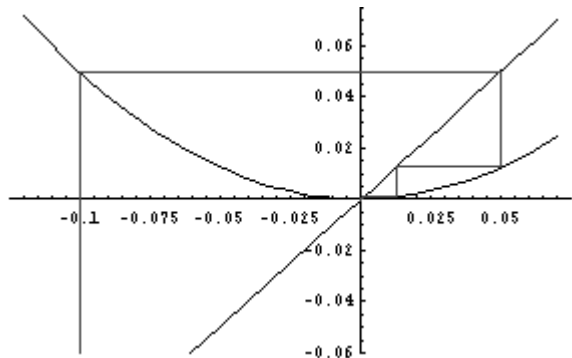


We can see that the zig-zag line moves toward the equilibrium point  $(0,0)$  (input value = equilibrium value = 0). This shows that if we start with an initial value slightly above the equilibrium value, then the sequence of values converges towards the equilibrium value. Next we create a Cobb-web diagram for an initial value slightly below the equilibrium value by adjusting the values of `xinit`, `xmin`, `xmax` and `focus` in the function `MapIt`. We can choose `xinit = -0.1`, `{xmin, xmax} = {-0.12, 0.02}` (equilibrium value at 0), and `focus = -0.05`.

```
MapIt[{5x^2, x}, -0.1, 5, {-0.12, 0.02}, -0.05, 1]
```



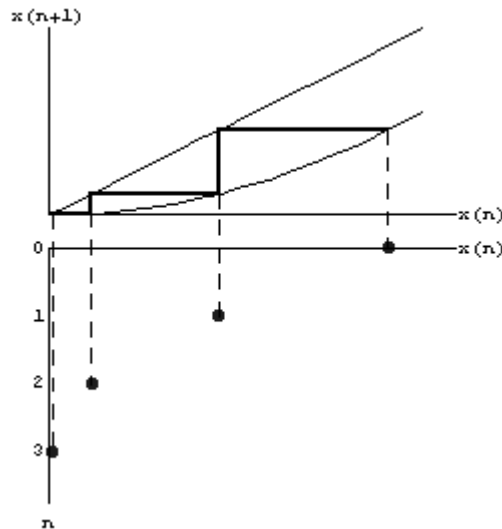
You may be a bit surprised by this graph. However, a close look at the values in the column for A2 (in the previous table) shows that  $x(2) = 0.05$ . All later values in the sequence are smaller. Thus, we should use `{xmin, xmax} = {-0.12, 0.07}` and `focus = (-0.12+0.07)/2 = -0.025` to see the full Cobb-web diagram. (Alternatively, you can choose `focus = 0` for simplicity).



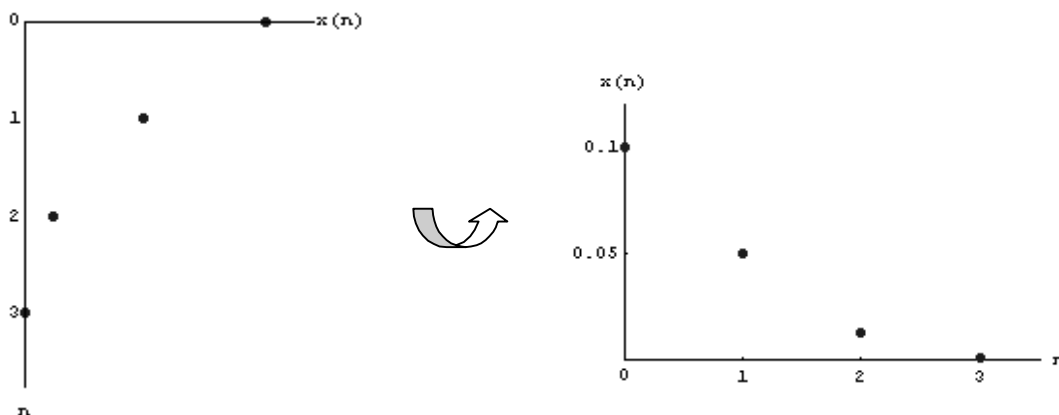
Since the sequence values approach the equilibrium value whether  $x(0)$  is slightly above or below the equilibrium value, we conclude that  $x = 0$  is a **stable** equilibrium.

The Cobb-web diagram looks quite different from the graphs we have seen before. Note that the axes display input and output values, but not time. Thus, the Cobb-web diagram visualizes the iterative model equation (where  $x(n+1)$  is a function of  $x(n)$ ), whereas the graphs created by **ListGraph** visualize the model solution (where  $x(n)$  is a function of  $n$ ).

Below is an illustration on how the two types of graphs relate to each other. We start by creating a Cobb-web diagram. Below this graph we put another set of horizontal and vertical axes. On the horizontal axis we mark  $x(n)$ , on the vertical axis we mark the time  $n$ . Now we extend the vertical lines of the Cobb-web graph downwards (through the value  $x(k)$ ) until we reach time  $k$ .

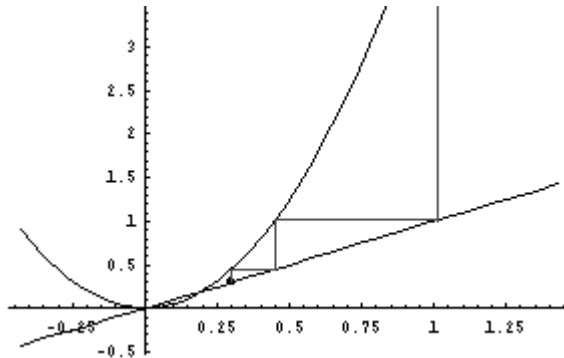


If we now take the lower part of this graph and rotate it by  $90^\circ$ , then we get a graph corresponding to the explicit solution, the type of graph that is produced by **ListGraph**.



Let's now check the second equilibrium  $x = \frac{1}{5} = 0.2$  with this new method, using initial values  $x(0) = 0.1$  and  $x(0) = 0.3$ . We have already seen the Cobb-web diagram for  $x(0) = 0.1$  and know that the values tend toward 0, not 0.2. This indicates that the equilibrium cannot be stable. Let's see what happens for  $x(0) = 0.3$ . (Note that focus was not chosen to be the midpoint.)

```
MapIt[{5x^2, x}, 0.3, 5, {0.15, 2}, 0.5, 1]
```



The graph shows that for a initial value slightly above the equilibrium value, the sequence again moves away from the equilibrium value of 0.2. This makes  $x = \frac{1}{5} = 0.2$  an **unstable** equilibrium.

### Activity 6.6.2

For the DDS given below, determine the stability of the equilibrium value(s) in two different ways, by using 1) the functions **IteratedValueSeq** and **ListGraph**, and 2) the function **MapIt**. (Note: These are the DDS from Activity 6.6.1)

- a)  $x(n+1) = x(n) \cdot (1 - x(n))$
- b)  $x(n+1) = x(n) - 2x(n)^2 + 3x(n)^3$
- c)  $x(n+1) = x(n)^2 + x(n)^3$

You can see a live animation of the Cobb-web diagram by using the function **LiveMap**. **LiveMap** uses entries very similar to the ones used in **MapIt**.

#### ?LiveMap

The function `LiveMap[{func, x}, xinit, n, {xmin, xmax}, {ymin, ymax}]` creates the graphics for an animation of  $n$  iterations of the function `func` expressed in terms of `x`. The iteration starts at  $x = xinit$  and is displayed for input values from `xmin` to `xmax` and output values from `ymin` to `ymax`.

**MapIt** and **LiveMap** only differ in the last two entries. In addition to specifying the range of display on the horizontal axis, we also indicate the range of display for the vertical axis. **LiveMap** creates a sequence of graphs, each displayed in a separate cell. Double clicking the cell bracket that encloses **all** of the graphs (2<sup>nd</sup> bracket from the left) collapses the cells so that only the first graph remains visible. Select **Cell -> Animate Selected Graphics** or use the shortcut Ctrl-Y to start the animation. You can control the speed of the animation by using the video panel that appears on the lower left corner of the notebook. To stop the animation temporarily, use the || button. To permanently stop the animation, click anywhere in the notebook outside the graph.