6.5 Equilibrium Values and Their Stability for First-Order Linear DDS

We have encountered the notion of a system in equilibrium in Section 6.2. In the credit card application, the equilibrium value represented a balance which did not change over time. For a first order linear DDS in equilibrium, the iterative model equation becomes

$$x(n) \underbrace{=}_{\text{equilibrium}} x(n+1) = a \cdot x(n) + b$$

Replacing x(*n*) by *x*, the above equation reduces to

$$x = a \cdot x + b \qquad (*)$$

where x denotes the equilibrium (output) value. If *a* and *b* are given, we can solve for *x*:

$$\begin{aligned} x &= a \cdot x + b \\ x - a \cdot x &= b \\ x(1-a) &= b \\ x &= \frac{b}{(1-a)} \quad \text{if } a \neq 1. \end{aligned}$$

If $a = 1$, then there is no solution unless $b = 0$ as equation (*) reduces to

$$\begin{aligned} x &= x + b \\ 0 &= b. \end{aligned}$$

If $b = 0$, then the value of *x* does not matter, i.e., any value is an equilibrium value. Altogether, we have:

Theorem 3 (Equilibrium values for First-Order Linear DDS)

The equilibrium value *x* for the first-order linear DDS

$$x(n+1) = a \cdot x(n) + b$$

is given by

$$x = \frac{b}{1-a} \qquad \text{if } a \neq 1$$

and

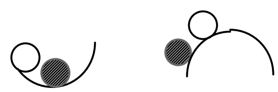
$$x = \text{any value} \quad \text{if } a = 1 \text{ and } b = 0.$$

In all other cases, no equilibrium value exists.

Remark: In the model of Section 6.1, the equilibrium value is always zero unless $a = 1$.

Equilibrium values are classified as either *stable* or *unstable*. The following graphical example from physics explains these classifications. Imagine you have two bowls, one upright and the other upside down. Place a ball into/onto each of these bowls. If you are careful, you can place the ball in such a way that it will not move unless it is given a small push. Since there is no movement, the balls are said to be in equilibrium. However, the two cases are distinguished by what happens if the balls are pushed slightly off the equilibrium position.

The left ball will roll back to its equilibrium position, whereas the ball on the right will move further and further away from its equilibrium position. Thus, the first equilibrium is called a *stable equilibrium*, and the second one is called an *unstable equilibrium*.

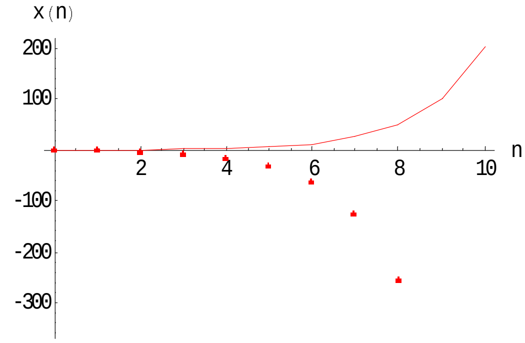


Definition:
An equilibrium value *x* is called *stable* (or *attracting*), if the sequence of output values tends towards *x*, for all initial values of $x(0)$ that are close to *x*. If the output values tend away from *x*, even though $x(0)$ is close to the equilibrium value, then the equilibrium is called *unstable* (or *repelling*). If the output values sometimes tend toward *x*, and for other initial values tend away from the equilibrium value, then the equilibrium is called *semi-stable*. In all other cases, the equilibrium is called *neutral*.

Let's check this definition using the two models of Activity 6.1.2, namely $x(n+1) = 2x(n)$ and $x(n+1) = -\frac{3}{4}x(n)$. In both models, the additive constant *b* is zero; thus, the equilibrium value is $x = 0$. To check for the type of equilibrium, we need to use starting values $x(0)$ that are close (on either side) to $x = 0$, the equilibrium value. Two of the initial values used in Activity 6.1.2, $x(0) = 0.2$ and $x(0) = -1$, respectively, satisfy this requirement. Below are the tables of values computed in Activity 6.1.2, together with a graph of the two sequences. For easier identification, the sequence of output values starting at $x(0) = 0.2$ are connected, whereas the values for the sequence starting at $x(0) = -1$ are not connected.

Model a): $x(n+1) = 2x(n)$

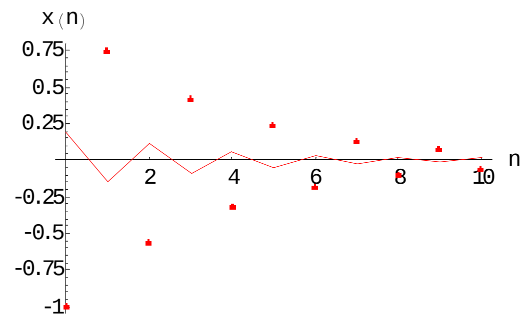
n	$x(n)$	$x(n)$
0	0.2	-1
1	0.4	-2
2	0.8	-4
3	1.6	-8
4	3.2	-16
5	6.4	-32
6	12.8	-64
7	25.6	-128
8	51.2	-256
9	102.4	-512
10	204.8	-1024



For both initial values, the output values of the respective sequences move further and further away from the equilibrium value of 0 (on both sides) $\Rightarrow$ **unstable equilibrium**.

Model b): $x(n+1) = -\frac{3}{4}x(n)$

n	$x(n)$	$x(n)$
0	0.2	-1
1	-0.15	0.75
2	0.1125	-0.5625
3	-0.0844	0.4219
4	0.0634	-0.3164
5	-0.0475	0.2373
6	0.0356	-0.1780
7	-0.0267	0.1335
8	0.0200	-0.1001
9	-0.0150	0.0751
10	0.0113	-0.0563



For each of the two initial values, the sequence of output values gets closer and closer to the equilibrium value x = 0. This is also evident in the table, where the output values approach zero $\Rightarrow$ **stable equilibrium**.

We can use the summary table about the long-term behavior of the model in Section 6.1 to determine for which values of the parameter a the equilibrium value is stable. Of the five cases considered in this table, we only need to look at cases where x(0) ≠ 0 (as the equilibrium value is 0). If a = 1, then by Theorem 3, every value is an equilibrium value, and since $x(n)$ = $x(0)$ always, every equilibrium is stable. The next two cases, a = -1 and $|a| > 1$, lead to a neutral and an unstable equilibrium, respectively. Finally, the case $|a| < 1$ results in a stable equilibrium.

A similar result can be proved for the case where $b \neq 0$ (Section 6.2) using methods from Calculus. Altogether, we have

Theorem 4 (Stability of Equilibrium for First-Order Linear DDS)
The equilibrium value *x* for the first-order linear DDS

$$x(n+1) = a \cdot x(n) + b$$

is **stable** if $|a| < 1$ and **unstable** if $|a| > 1$. If $a = 1$(and b = 0), then every value is a **stable** equilibrium; if $a = -1$, then the equilibrium is **neutral**.

We can illustrate Theorem 4 for models with $b \neq 0$ using the credit card example from Activity 6.2.2. The model was given by

$$x(n+1) = 1.01075 \cdot x(n) - 100.$$

Thus, the equilibrium value is ($a = 1.01075, b = -100$)

$$x = \frac{b}{1-a} = \frac{-100}{1-1.01075} = 9{,}302.33$$

If we graph the sequence of output values for initial balances of \$9,250 and \$9,350 we see that they move away from the equilibrium; hence, the equilibrium is unstable. This checks out with the theorem, as $a = 1.01075 > 1$.