6.4 Explicit Solution for First-Order Linear DDS

In Section 6.1, we derived the explicit solution (= explicit formula for $x(n)$ in terms of n) for the special case of the linear first-order DDS where $b = 0$. We computed $x(1)$, $x(2)$,... by using the iterative equation repeatedly. Each time, we substituted the previous answer into the iterative model equation. From the resulting expressions, we guessed a pattern and then verified it by substituting the pattern into the iterative relation. We will use the very same approach now for the general case, where b can be any value.

We start with the iterative model equation:

$$x(n+1) = a \cdot x(n) + b$$

This leads to:

$n = 1$: $\quad x(1) = a \cdot x(0) + b$

$n = 2$:
$$\begin{aligned} x(2) &= a \cdot x(1) + b \\ &= a \cdot [a \cdot x(0) + b] + b && \text{substitute } x(1) \\ &= a^2 \cdot x(0) + ab + b && \text{expand} \end{aligned}$$

$n = 3$:
$$\begin{aligned} x(3) &= a \cdot x(2) + b \\ &= a \cdot [a^2 \cdot x(0) + ab + b] + b && \text{substitute } x(2) \\ &= a^3 \cdot x(0) + a^2 b + ab + b && \text{expand} \end{aligned}$$

$n = 4$:
$$\begin{aligned} x(4) &= a \cdot x(3) + b \\ &= a \cdot [a^3 \cdot x(0) + a^2 b + ab + b] + b && \text{substitute } x(3) \\ &= a^4 \cdot x(0) + a^3 b + a^2 b + ab + b && \text{expand} \end{aligned}$$

The pattern that emerges is

$$\begin{aligned} x(n) &= a^n \cdot x(0) + a^{n-1} b + a^{n-2} b + \cdots + ab + b \\ &= a^n \cdot x(0) + b[a^{n-1} + a^{n-2} + \cdots + a + 1] \end{aligned}$$

Let's verify our guess by substituting it into the iterative model equation:

$$
\begin{aligned}
x(n+1) &= a \cdot x(n) + b \\
&= a \cdot [a^n \cdot x(0) + a^{n-1}b + a^{n-2}b + \cdots + ab + b] + b && \text{substitute } x(n) \\
&= a^{n+1} \cdot x(0) + a^n b + a^{n-1}b + \cdots + a^2 b + ab + b && \text{distribute terms} \\
&= a^{n+1} \cdot x(0) + b[a^n + a^{n-1} + \cdots + a^2 + a + 1] && \text{simplify}
\end{aligned}
$$

We have established that the explicit solution is given by

$$x(n) = a^n \cdot x(0) + b[a^{n-1} + a^{n-2} + \cdots + a + 1]$$

This expression can be simplified, using the fact that the sum inside the brackets is a *geometric sum*. It can be shown that

$$a^{n-1} + a^{n-2} + \cdots + a + 1 = \begin{cases} \dfrac{1-a^n}{1-a} & \text{if } a \neq 1 \\ n & \text{if } a = 1 \end{cases}.$$

We can summarize this result in the following theorem:

Theorem 2 (Explicit Solution for First-Order Linear DDS)

The explicit solution for the first-order linear DDS of the form

$$x(n+1) = a \cdot x(n) + b$$

is given by

$$x(n) = \begin{cases} a^n x(0) + b\dfrac{1-a^n}{1-a} & \text{if } a \neq 1 \\ a^n x(0) + b \cdot n & \text{if } a = 1 \end{cases}$$

<u>Remark</u>: Note that Theorem 1 in Section 6.1 is a special case of this theorem for $b = 0$.

We will now look at several examples for which we can use the result of Theorem 2. In each case, we start by identifying the meaning of the input and output variables. Next, we translate the verbal description of the problem into an iterative model equation, using the paradigm new = old + change. Once we have derived the iterative model equation, we identify the values for the parameters (= constants) *a* and *b*, and then utilize the explicit solution to answer the question at hand. Here is an example of the process.

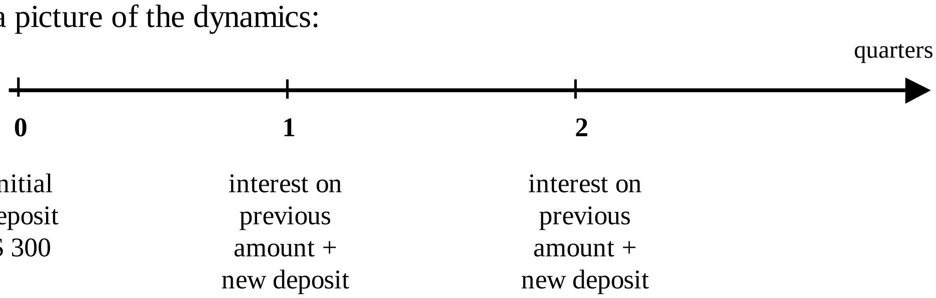
Example 1

Suppose you start a savings account which pays 5% interest compounded quarterly (i.e., you are paid interest 4 times a year, every three months, and receive $\frac{1}{4}$ of 5% = 1.25% at the end of each quarter). You initially deposit $300 into the account, and then save $300 at the beginning of each quarter. How much money do you have in your account after 5 years?

The first step is to identify the meaning of the input and output variables. We are interested in the account balance at a particular time. Therefore, time is the input variable, and the account balance is the output variable. The account balance is measured in dollars, and time is measured in months. However, changes in account balance occur only at the beginning of every quarter (= 3 months), when we save an additional $300 and receive the interest payment. Thus, time should be measured in three-month segments, or quarters. Overall, we have

$$\begin{aligned} n &= \#\text{ of quarters after initial deposit} \\ B(n) &= \text{balance (in \$) after } n \text{ quarters} \\ B(0) &= \text{initial deposit} = \$\,300 \end{aligned}$$

Here is a picture of the dynamics:



Using the paradigm new = old + change, we get

$$\begin{aligned} B(n+1) &= B(n) + (\text{interest on } B(n) + \text{new deposit}) \\ &= B(n) + 0.0125B(n) + 300 \\ &= 1.0125B(n) + 300 \end{aligned}$$

Comparing our model equation with the expression $x(n+1) = a \cdot x(n) + b$, we read off a = 1.0125 and b = 300. Substituting these values for *a* and *b*, the explicit solution as given by Theorem 2 is:

$$B(n) = (1.0125)^n \cdot B(0) + 300\left(\frac{1-(1.0125)^n}{1-1.0125}\right)$$

We want the balance after 5 years = $5 \cdot 4 = 20$ quarters.

$$\begin{aligned} B(20) &= 1.0125^{20} \cdot 300 + 300 \cdot \left(\frac{1-1.0125^{20}}{1-1.0125}\right) \\ &= 7153.50. \end{aligned}$$

The balance after 5 years will be 7153.50. (Notice that this balance also includes a deposit of \$300 at time $n = 20.$)

Here is another problem that shows up frequently and is closely related to the notion of half-life (discussed in Section 6.1).

Example 2

How many years does it take to double your money if you invest it at 5%, compounded monthly?

Again, the input variable is time, measured here in months (as interest is paid every month). The output variable is the account balance

$$\begin{aligned} n &= \text{\# of months (after initial investment)} \\ B(n) &= \text{balance (in \$) after } n \text{ months} \\ B(0) &= M \text{ (not given)} \end{aligned}$$

The problem does not indicate the initial amount, and it will turn out that the value of $B(0)$ is irrelevant. However, you can substitute your favorite amount, say \$1000, if you do not like the abstract number M.

The question here is to find a time, rather than an account balance. Let's first set up the model equation and see how we can proceed from there. Interest is paid at 5%, compounded **monthly**, but the interest rate is given per **year**, so each month you get just $\frac{1}{12}$ of that, i.e., $\frac{5}{12}\% = 0.4167\%$. (In general, if you get *interest compounded k times a year*, and the interest rate is r%, then each time you are paid interest, you get $\frac{r}{k}\%$.)

The only change in the balance is the monthly interest payment, so we have

$$\begin{aligned} \text{new} &= \text{old} + \text{change} \\ B(n+1) &= B(n) + 0.004167B(n) \\ &= 1.004167B(n) \end{aligned}$$

This is an iterative model equation of the type discussed in Section 6.1, with a = 1.004167. Therefore, the explicit solution is given by

$$B(n) = a^n B(0) = (1.004167)^n B(0)$$

We know the balance to be achieved is double the original amount, or $B(n) = 2B(0)$. Substituting this into the left side of the equation yields

$$2 \cdot B(0) = 1.004167^n B(0)$$

Canceling $B(0)$ on both sides of the equation (which is possible since we know $B(0) \neq 0$ - we had to start with some money in order to double), we get

$$2 = 1.004167^n$$

To solve for *n*, the time at which doubling occurs, we can use the palette function **SolveIt**:

```
SolveIt[2, 1.004167^n, n]
```

If you want to use an initial guess, you could use $20 \cdot 12 = 240$. (If no interest was paid on the interest, then it would take 20 years until doubling: $20\,\text{years} \cdot 5\% = 100\%$; however, since the interest becomes part of the principal, we should reach doubling in a shorter period of time.)

```
SolveIt[2, 1.004167^n, n,240]
{{n→166.688}}
```

Thus, after **167 months** = 13 years and 11 months, **the initial amount will have doubled**. Note that we have rounded up the value of *n*, as only after that payment will the initial amount have increased to twice as much. Like before, we can use methods for solving exponential equations to solve the above equation analytically:

$$n = \frac{\ln(2)}{\ln(1.004167)}$$

where ln, again, stands for the natural logarithm. You can check the previous result using the built-in *Mathematica* function **Log**.

The next example deals with a common problem, namely, deciding how much money has to be saved to reach a given amount at a specific time.

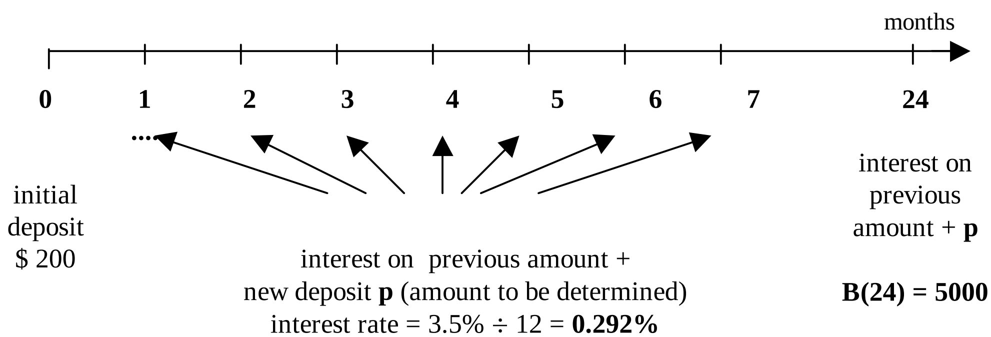
Example 3

You just had your 16th birthday, and your grandparents gave you \$200. You decide to take this money and start a savings account so that you can buy a car upon your 18th birthday. You have checked around for used cars, and decide that \$5000 will buy you a good used car. The bank currently pays 3.5% interest, compounded monthly. How much money do you have to save each month in order to achieve your goal?

The input variable time is measured in months (2 years = 24 months), and the output variable is the balance in dollars

$$\begin{aligned} n &= \text{\# of months (after your } 16^{\text{th}} \text{ birthday)} \\ B(n) &= \text{balance (in \$) after } n \text{ months} \\ B(0) &= 200 \\ B(24) &= 5000 \end{aligned}$$

Let's visualize the account activity (= change of balance) with a timeline:



The iterative model equation is given by

$$\begin{aligned} B(n) &= B(n-1) + \left(\tfrac{3.5}{12}\%\right) \cdot B(n-1) + p \\ &= B(n-1) + 0.00292 \cdot B(n-1) + p \\ &= 1.00292 \cdot B(n-1) + p \end{aligned}$$

Thus, $a = 1.00292$ and $b = p$ (the desired payment amount). The explicit solution is given by

$$B(n) = (1.00292)^n B(0) + p \cdot \left(\frac{1-(1.00292)^n}{1-1.00292} \right).$$

We know the balance at time n = 24. By substituting the initial and final balances, and also $n = 24$, we get the following equation:

$$B(24) = 5000 = (1.00292)^{24} \cdot 200 + p \cdot \left(\frac{1-(1.00292)^{24}}{1-1.00292} \right)$$
$$5000 = 212.01 + p \cdot 20.5646$$

Now, we can solve analytically for *p*:

$5000 - 212.01$	$=$	$p \cdot 20.5646$	subtract
4787.99	$=$	$p \cdot 20.5646$	simplify
$\frac{4787.99}{20.5646} = 232.827$	$=$	p	divide and simplify

Alternatively, we can use **SolveIt**:

```
SolveIt[5000,212.01+ p*20.5646, p]
{{p→232.827}}
```

This means that **you should save $232.83 every month** to reach your goal of $5000 in two years.

In the following activity you will solve problems similar to the examples. You may want to draw a timeline to help you determine the relevant parameters for the model.

Activity 6.4.1

For each of the problems below, identify the input and output variables (with their units), set up the iterative model equation, and use the explicit solution to answer the question.

1. How many years does it take to double your money if you invest it at
   a) 7%, compounded monthly?
   b) 7.5%, compounded quarterly?
   c) 8%, compounded semiannually (twice a year)?

2. You just became a proud grandparent. Since you think education is important, you want to start a savings account for your grandchild, to ensure that a college education is financially feasible. Currently, a four-year degree at a good private college costs in the $100,000 range. Without accounting for inflation, you set the goal of putting aside an amount of money every month that will grow to $100,000 in 18 years with the current interest rate of 4%, compounded monthly. How much money do you have to save each month?

3. At retirement, your savings account has a balance of $300,000 which is collecting 6% interest, compounded monthly. How much money can you withdraw each month so that your savings will last you

   a) 20 years b) 30 years.

4. Suppose you wish to buy house, but can only afford to pay $1,200 per month for a loan. You inquire with a mortgage broker who tells you s/he can get you a loan at 7.5% (compounded monthly) for 30 years. What is the largest loan amount can you afford? (Hint: You start with a debt balance equal to the size of the loan amount, which is reduced by your monthly payments to $0 at the end of the 30-year loan period.)