6.3 Classification of Discrete Dynamical Systems

The two models in Sections 6.1 and 6.2 are examples of discrete dynamical systems (DDS). Recall that the adjective “discrete” specifies how time is measured (see Section 5.2), whereas “dynamical” reflects the fact that the output values change over time.

One way to classify a DDS is by order. The order of a DDS refers to the range of dependency of the iterative model equation. In all our examples, the value of $x(n+1)$ was dependent only on the value of $x(n)$, i.e., the range of dependency was **one** time unit. Such a system is called a *first-order* system. Likewise, the range of dependency of a *second-order* system is **two** time units, i.e., $x(n+1)$ will depend on $x(n)$ and $x(n-1)$, the two most recent system values. Note that the range is determined by the “oldest” output value, whether or not all the intermediate output values are part of the iterative model equation.

The second way to classify a DDS is by linearity. If the model function (= right hand side of the iterative model equation) is a linear function of $x(n)$, then the DDS is call *linear*. The examples in Sections 6.1 and 6.2 are both linear systems:

$$x(n+1) = a \cdot x(n) \qquad \text{(Section 6.1)}$$
$$x(n+1) = a \cdot x(n) + b \qquad \text{(Section 6.2)}$$

The first example is a special case of the second one (for $b = 0$). The iterative model equation is of the form

$$f(x(n)) = a \cdot x(n) + b,$$

which can be identified as a linear function by replacing $x(n)$ by x. Thus, the iterative model equation of a *first-order linear DDS in one variable* is given by

$$x(n+1) = a \cdot x(n) + b\,.$$

For linear systems of higher order, the rule for linearity can be described as follows:

> A DDS is linear if all of the output values on the right hand side of the iterative model equation have either power zero or one, and are multiplied by constants only.

Below are examples of both linear and nonlinear systems of different orders.

a) $x(n+1) = 2x(n) - x(n-1) + 3$

b) $x(n+1) = \dfrac{x(n)}{1 - x(n)}$

c) $x(n+1) = x(n-2)$

d) $x(n+1) = 3x(n)^2$

e) $x(n+1) = \sqrt{x(n)}$

Let's identify the order of these systems and determine whether or not they are linear.

<u>Solution</u>:

a) This is a linear system of second order (since the dependency range is two).
b) This is a first-order system which is not linear (because of the division by 1- $x(n)$).
c) This is a linear system of third order (since the dependency range is 3; recall that $x(n)$ and $x(n\text{-}1)$ do not have to be part of the equation).
d) This is a first-order system which is not linear (because $x(n)$ is squared).
e) This is a first-order system which is not linear (because the square root function corresponds to power $\frac{1}{2}$).

Activity 6.3.1

Classify the following discrete dynamical systems as linear or nonlinear, and determine their order. Give reasons for your answers.

a) $x(n+1) = 2 - x(n-1)$

b) $x(n+1) = -3x(n) + 1$

c) $x(n+1) = (x(n) - 2) \cdot x(n) + 4$

d) $x(n+1) = x(n) - 2x(n-1)^3 + 4$