

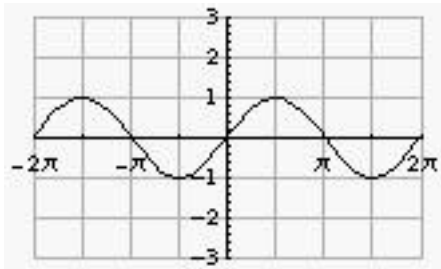
3.6 Sine and Cosine Functions

The general form of a *sine* or *cosine* function is given by:

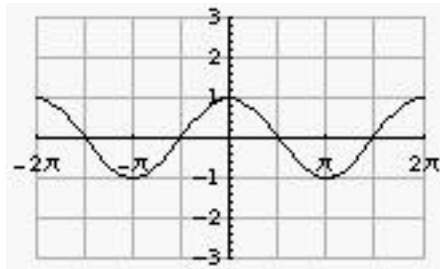
$$f(x) = a\sin(bx + c) + d \quad \text{and} \quad f(x) = a\cos(bx + c) + d$$

where a , b , c , and d are constants with $a \neq 0$, and x is measured in radians. (π radians = 180° , the angle measure of a half circle.) Their graphs look like a wave, with one wave segment repeated continuously. The simplest versions of these functions are $f(x) = \sin(x)$ and $f(x) = \cos(x)$ in the case where $a = b = 1$ and $c = d = 0$.

$$f(x) = \sin(x)$$



$$f(x) = \cos(x)$$

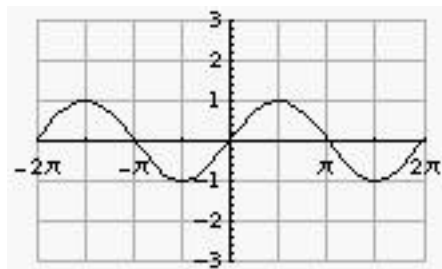


Notice that the graphs look very similar, except the cosine function is moved to the left by an amount of $\frac{\pi}{2}$. Indeed, every cosine function is a shifted sine function, and therefore, we will concentrate on sine functions only from now on.

Activity 3.6.1

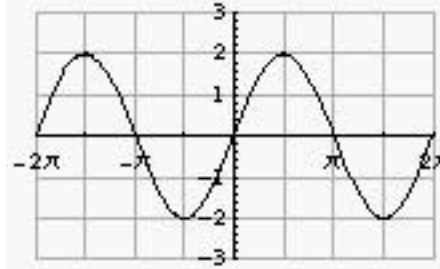
For the functions given below, identify the values of the constants a , b , c , and d .

1) $f(x) = \sin(x)$



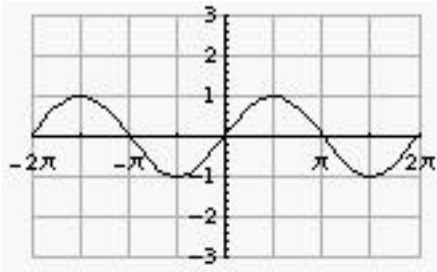
$$a = \quad b = \quad c = \quad d =$$

$f(x) = 2\sin(x)$



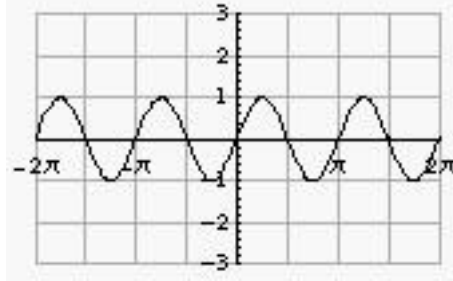
$$a = \quad b = \quad c = \quad d =$$

2) $f(x) = \sin(x)$



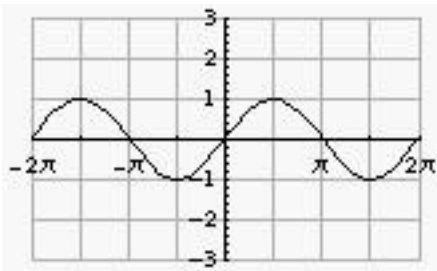
$$a = \quad b = \quad c = \quad d =$$

$f(x) = \sin(2x)$



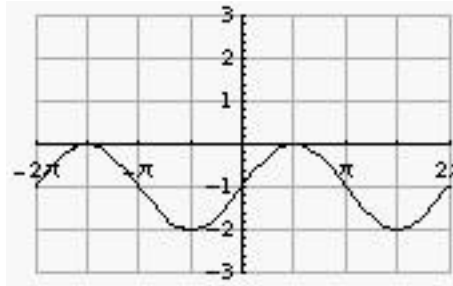
$$a = \quad b = \quad c = \quad d =$$

3) $f(x) = \sin(x)$



$$a = \quad b = \quad c = \quad d =$$

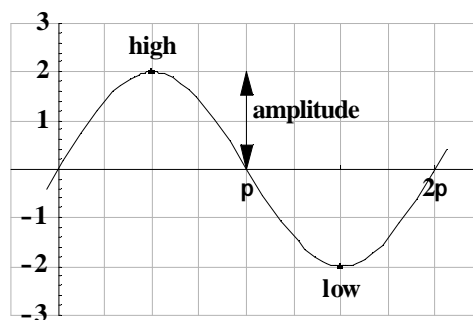
$f(x) = \sin(x) - 1$



$$a = \quad b = \quad c = \quad d =$$

Looking at the pairs of graphs for examples 1), 2), and 3), notice that only one of the constants is different. This means that any change in the graph is caused by that particular constant. Let's consider each pair in detail:

- Here the constant a changed from 1 to 2. The graph now reaches twice as high and twice as low. This property is characterized by the *amplitude*. The amplitude of a sine/cosine function is given by $|a|$. To read off the amplitude from the data, compute the **difference** between the maximal and minimal **output** values (to measure vertical distance) and divide by 2. In the example given below, the amplitude is computed as $a = (2 - (-2)) \div 2 = 2$.

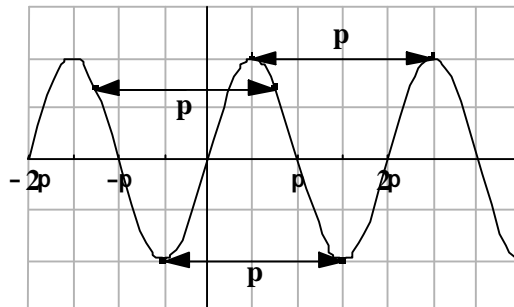


Thus, the amplitude gives half the amount of the overall variation in output values and measures how much the extreme high and low values differ from the average value.

- 2) In this case, the constant b has changed from 1 to 2. If we compare the graphs for input values from 0 to $2p$, on the right graph we see two repetitions of the basic segment (= one full wave). Therefore, b indicates the *frequency* of a sine/cosine curve, i.e., the number of repeated segments in an interval of length $2p$. In applications, we are often interested in a closely related value, the *period* p , which gives the length of the repeated segment or cycle. Since b equals the number of repeated segments in an interval of length $2p$, we have

$$\underbrace{b}_{\text{\# of segments}} \cdot \underbrace{p}_{\text{length of segment}} = 2\pi$$

This formula provides an easy way to compute the period as we can easily read off the value of b from the functional expression. If we only have data, then we can read off the period as follows: Select two consecutive points at the same output level where the function is either both increasing or decreasing. (Simple points to consider are two consecutive high or low points.) The **horizontal** distance (= difference in input values) between these points gives the period. In the graph below, the period is $2p$.



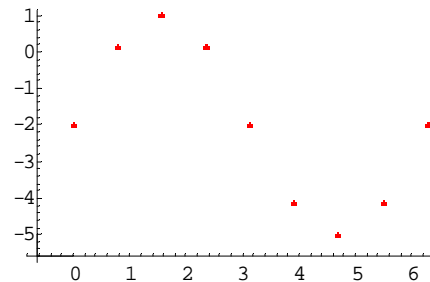
- 3) Here, the only constant that has changed is d , from 0 to -1 . The effect in the graph is a vertical shift downward by 1 unit. Thus, d gives the *vertical shift*; negative values correspond to a downward shift while positive values correspond to an upward shift. A second interpretation of the constant d is the *average* or *middle value* for the sine function. To find the value of d from the graph, we **add** the maximal and minimal output values, then divide by 2. From the second graph of 3) in Activity 3.6.1, we see that the biggest output value is 0 and the lowest one is -2 . Thus, $d = (0 + (-2)) \div 2 = -1$.

The remaining constant in the formula for the sine function is c . This constant influences the horizontal shift of the sine function, but can not always be read off directly from the graph.

However, in practical applications, the values of the other three constants are more important. Furthermore, the horizontal shift is not uniquely determined because of the repetitive nature of the sine function.

Here is an example on how to read off the relevant parameters for the data given below:

x	$f(x)$
0	-2
0.785	0.121
1.571	1
2.356	0.121
3.142	-2
3.927	-4.121
4.712	-5
5.498	-4.121
6.283	-2



The graph of the data shows a wavelike shape. If we assume for the moment that this behavior will continue, we can use the graph and the table to determine the values of the constants a , b and d of a sine function.

The amplitude a gives the amount by which the extreme values differ from the average value. We compute a as half the difference between maximal and minimal output value. From the table we see that the largest output value (in the $f(x)$ column) is 1, and the smallest output value is -5 . Thus, the amplitude is

$$a = \frac{(1 - (-5))}{2} = \frac{(1 + 5)}{2} = 3$$

To compute the value of the frequency b , we read off the period p (= length of repeated segment) and then use the formula connecting frequency and period. Recall that we are looking for points at which the output values are the same (or approximately the same), and the behavior of the function is identical (either increasing or decreasing values from left to right). Thus, even though the output values are the same for the two points (0.785, 0.121) and (2.356, 0.121), the behavior of the function near those two points is different. For the first point, the function increases, while for the second point, the function decreases. The only points that can be used are (0, -2) and (6.283, -2). The period is the horizontal distance between these points, computed as the difference in input values. Thus, we have:

$$\text{period } p = 6.283 - 0 = 6.283$$

and therefore, (since $b \cdot p = 2\pi$)

$$b = \frac{2\pi}{p} = \frac{6.283}{6.283} = 1.$$

Last, but not least, we compute d , the average value (or vertical shift). To compute d we add the maximal and minimal output values, then divide by 2:

$$d = \frac{(1 + (-5))}{2} = \frac{-4}{2} = -2.$$

Remark: Be careful not to confuse the computation of amplitude and average value. Both involve maximal and minimal output values, but for the amplitude the values are subtracted, whereas for the average value they are added, before being divided by 2.

Activity 3.6.2

For the given data, determine from the graph whether the data follows a sine function (giving reasons for your answer). If a sine function is likely, compute the amplitude, period, and vertical shift for the function. Indicate which points you have selected to compute these three quantities and use the table of values to do the necessary computations.

x	$f(x)$
-4	4.110732
-3.5	3.640774
-3	1.937131
-2.5	-1.07968
-2	-4.20489
-1.5	-5.31053
-1	-5.6338
-0.5	-4.33227
0	-1
0.5	2.272046
1	3.708389
1.5	4.468344
2	2.325158
2.5	-0.92528
3	-3.92069
3.5	-6.28067
4	-6.03912

