

### 3. Special Types of Functions

In this chapter, we will look at selected types of functions and their properties. We will build a “library” of shapes and characteristics of functions to help us decide what type of function may fit given data. The idea behind fitting a particular function to data is that we assume a functional connection between the input and output of the data that we observe. The goal is to uncover a function rule which then can be used to predict future values.

Here is an example which we will revisit later in the chapter. The data below gives the estimated median age of the first marriage of women in the United States. For example, in 1960, the median age was 20.3 years. This means that half of the women married (for the first time) before they were 20.3 years old.

Year	1960	1965	1970	1975	1980	1985	1990	1995
Median Age	20.3	20.6	20.8	21.1	22.0	23.3	23.9	24.5

Using this data we would like to answer questions such as:

- What will the median age of first marriages be in the year 2010?
- When will the median age of first marriages reach 30 years?

In order to find the “correct” underlying function, we look at the shape indicated by the graph of the data. If more than one function type matches the shape of the graph, then we can use other characteristics of these function types to select the best choice. In Chapter 4, we will see how we can find a specific function (of a given type) that best fits the data and how to compare the best fitting functions of different types.

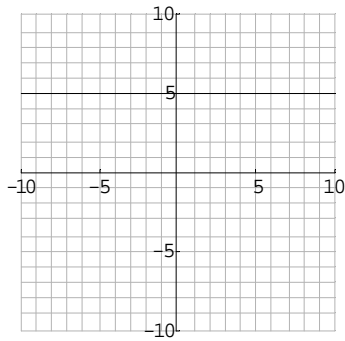
#### 3.1 Linear Functions

The graphs of *linear functions* are straight lines. Their functional description is as follows:

$$f(x) = a \cdot x + b$$

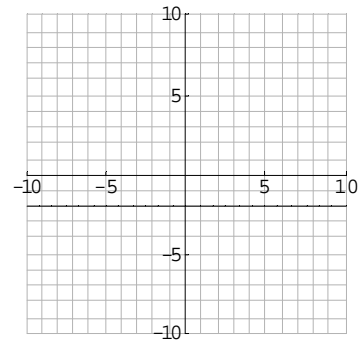
where  $a$  and  $b$  are constants. We will look at some examples and derive how the values of  $a$  and  $b$  influence the graph of the function. We start by identifying the values of  $a$  and  $b$  in the examples below. ( $a$  is the constant that is multiplied by the independent variable, and  $b$  is the additive constant.)

1)  $f(x) = 5$



$a = 0$        $b = 5$

$f(x) = -2$



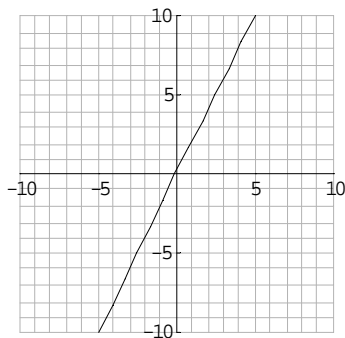
$a = 0$        $b = -2$

In both cases, the functional expression does not contain  $x$ ; hence, the multiplicative constant  $a$  must have been zero. The remaining constant is the value of  $b$ . Note that you can always check your answer by substituting your values for  $a$  and  $b$  into the general expression  $f(x) = a \cdot x + b$ .

Check:  $a = 0, b = 5 \Rightarrow f(x) = 0 \cdot x + 5 = 5$

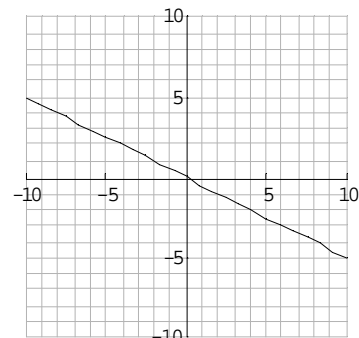
$a = 0, b = -2 \Rightarrow f(x) = 0 \cdot x + (-2) = -2$

2)  $f(x) = 2x$



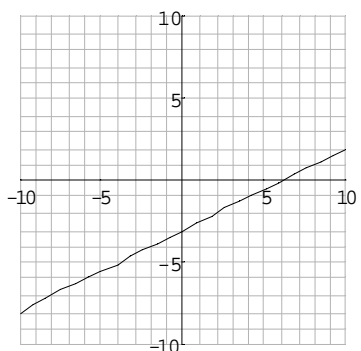
$a =$        $b =$

$f(x) = -\frac{1}{2}x$



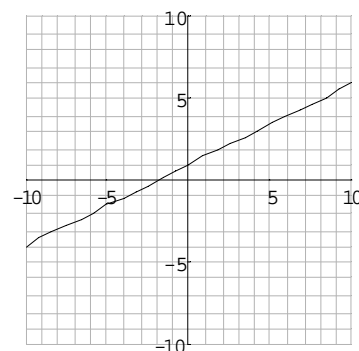
$a =$        $b =$

3)  $f(x) = \frac{1}{2}x - 3$



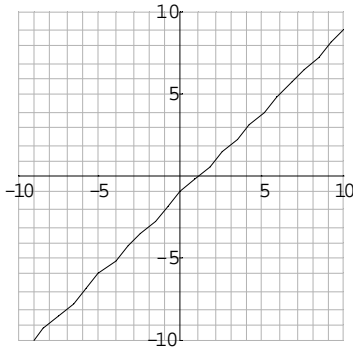
$a =$        $b =$

$f(x) = \frac{1}{2}x + 1$



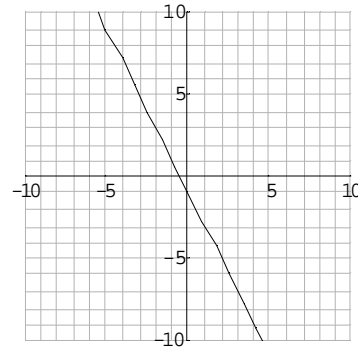
$a =$        $b =$

4)  $f(x) = x - 1$



$a =$        $b =$

$f(x) = -2x - 1$



$a =$        $b =$

**Activity 3.1.1**

For examples 2), 3), and 4), identify the values of the constants  $a$  and  $b$ . Check whether the values you found are correct by substituting them into the expression  $f(x) = a \cdot x + b$ .

The values of  $a$  and  $b$  influence the position of the straight line. Furthermore, they have a specific meaning in the context of a given problem.

**Activity 3.1.2**

- In Example 1), the value of  $a$  is zero. What do the two graphs have in common?
- In Example 2), the value of the constant  $b$  is zero. What do the two graphs have in common?
- Two of the functions above have a negative value for  $a$ . What is special about the graphs of these functions?

If we look at Example 3) we see that the value of  $a$  is the same for the two functions, and we notice that the steepness of the two straight lines is the same. Similarly, in Example 4), the two functions have the same value for the constant  $b$ . In this case, their common characteristic is that the graphs intersect the vertical axis at the same point, namely  $(0, b)$ .

It can be shown in general that the shape of the graph of a linear function is determined by the value of the *slope*,  $a$ , and the intercept with the vertical axis located at the point  $(0, b)$ . The slope

$a$  indicates how much the output value changes for every **one unit increase** of the input independent of the current input value. Linear functions are the only functions with this property.

The slope often has an important interpretation in the context of the problem. The next example will show how to compute the slope or unit increase and will illustrate its interpretation.

**Example:**

When selling your car you can check how it compares against the “average” car as defined by the AAA bluebook. This bluebook lists items such as value of extras and the average number of miles **per year**. Depending on whether your car has less or more miles than this average, its value will either increase or decrease.

Let’s say you have been driving the used car you bought for a number of years and want to find out how many miles per year you added to the car (on average). When you bought the car, it was 8 years old and had 45,000 miles. Now the car is 19 years old and has 143,000 miles on it. How many miles per year did you add?

First, compute how many miles you added:

$$143,000 - 45,000 = 98,000 .$$

This increase occurred over  $19 - 8 = 11$  years; thus, you added

$$\frac{98,000}{11} = 8,909.9 \approx 8,910 \text{ miles per year.}$$

Remark: 1) The unit increase, or slope, is the number of miles (output) **per** year (input).

- 2) If we assume that every year a **fixed** number of miles is added (=constant slope) then the function that gives the number of miles driven by a certain age of the car will be a **linear** function.

The method used in the example for computing the number of miles per year can be described as follows: Divide the difference in output (=mileage in the example) by the difference in input (=years in the example). In mathematical notation this becomes

For a linear function and any two points  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$ , the slope is given by the *difference quotient* or *first unit difference*

$$a = \frac{f(x_2) - f(x_1)}{x_2 - x_1} .$$

**Remark:** 1) In the example above, the two points would have been (8, 45000) and (19, 143000), and the yearly increase (=slope) is given by

$$\frac{143000 - 45000}{19 - 8} = \frac{98000}{11} \approx 8,910.$$

2) It is not important in which order the points are taken, as long as the order of the components is the same in numerator and denominator. For example, we can switch the order of the points used above:

(19, 143000) and (8, 45000)

and we still get the same difference quotient

$$\text{per year increase} = \frac{45000 - 143000}{8 - 19} = \frac{-98000}{-11} \approx 8,910.$$

### Activity 3.1.3

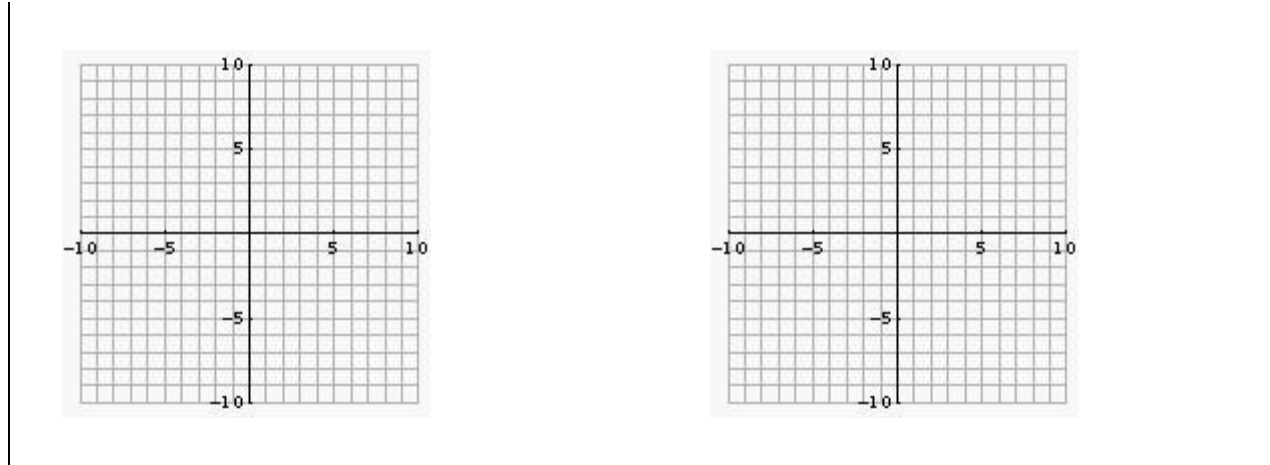
For the two linear functions below, identify the values of  $a$  and  $b$  (as in Activity 3.1.1). Then compute the table of values, and the slope for three different pairs of input and output values. (Hint: The slope should be the same for pairs from the same function since linear functions have a constant slope, and the resulting value of the slope should equal the value of  $a$ .) Finally, graph the two functions in the coordinate system.

$$f(x) = -x + 2 \quad a = \quad b =$$

$$f(x) = \frac{3}{2}x + \frac{1}{2} \quad a = \quad b =$$

$x$	$f(x)$	slope
-2	4	$\frac{4-3}{(-2)-(-1)} = \frac{1}{-1} = -1$
-1	3	
0		
1		
2		
3		

$x$	$f(x)$	slope



Thus, in summary, we have

#### Properties of Linear Functions

- 1) The functional expression of a linear function is given by  $f(x) = a \cdot x + b$ .
- 2) A linear function has constant slope,  $a$ . The value of  $a$  gives the fixed rate of increase (if  $a$  is positive) or decrease (if  $a$  is negative).
- 3) The graph of a linear function is a straight line.

Now we will use this knowledge about linear functions to help us determine whether a set of data follows a linear function. When we are dealing with observed data, we usually do not have a functional expression (=formula), but just a table of values or a graph. To check for an underlying linear relationship between the input and output values, we check on the numerical and graphical properties of functions.

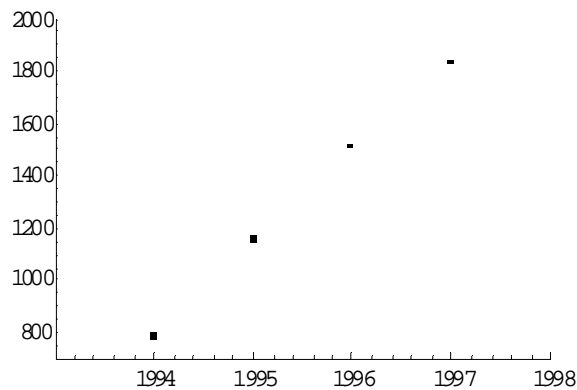
From the data, we can create a graph and check whether the data points lie (approximately) on a straight line. If this is not the case, then the linear relationship is not possible. On the other hand, if the graph passes the test for a linear function, then we can follow up with computing first unit differences for consecutive pairs of input and output values. If the data follows a linear function exactly, then those first unit differences would all be identical. Since most data contains slight errors, we would expect the first unit differences to be only approximately the same. We will illustrate this approach in the following example.

**Example:**

The table below gives the yearly market value (in \$ billion) for the NASDAQ from 1994-1997<sup>1</sup>. We will graph the function and compute the first unit differences (~slope) for the data to determine whether there is a linear relation between input (= year) and output (= market value).

Year	Market Value (in \$ billion)
1994	786.5
1995	1,159.9
1996	1,517.1
1997	1,834.8

We have two tools: the numerical one (computing the first unit differences) and the graphical one (checking for a straight line). As a first step, we graph the data. If the graph supports the notion that the data comes from a linear function by showing approximately a straight line, then we can use the first unit differences to tell us what the slope would be and further substantiate the hypothesis of an underlying linear function. If, on the other hand, the graph does not resemble a straight line, then we do not have to go through the computations for first unit differences. Let's start with a graph of the data.



This looks roughly like a straight line (try to put a pencil along the four points). Thus, it makes sense to compute the first unit differences to numerically verify that the data comes from a linear function. We take the first two values, (1994, 786.5) and (1995, 1159.9) as  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$ . Recall that we can compute the slope in two different orders. Here we adopt the order  $\frac{f(x_1) - f(x_2)}{x_1 - x_2}$ , as this makes it easy to work within the table (the order of pairs corresponds to moving downward in the table).

<sup>1</sup> <http://www.nasd.com/mr3b.html> (NASDAQ web site)

$$\Delta = \text{difference} = \frac{786.5 - 1,159.9}{1994 - 1995} = \frac{-373.4}{-1} = 373.4$$

Next, we take the second and third data values, (1995, 1159.9) and (1996, 1517.1):

$$\Delta = \frac{1,159.9 - 1,517.1}{1995 - 1996} = \frac{-357.2}{-1} = 357.2$$

The third and fourth data points, (1996, 1517.1) and (1997, 1834.8) give the following difference:

$$\Delta = \frac{1,517.1 - 1,834.8}{1996 - 1997} = \frac{-317.7}{-1} = 317.7$$

Thus, we can put together the following table:

x	f(x)	$\Delta$
1994	786.5	373.4
1995	1,159.9	
1996	1,517.1	357.2
1997	1,834.8	317.7

- Remarks: 1) Note that the number of first unit differences is one less than the number of data points.
- 2)  $\Delta$  is pronounced “delta” and is the Greek letter for d. It is commonly used to indicate a difference of values.

We now need to decide whether we believe in an underlying linear relationship. Let’s look at the values 373.4, 367.2 and 317.7. They are roughly equal; however, the difference between the biggest and smallest values is about 50, which is approximately  $\frac{1}{7}$  of the biggest unit difference. This is somewhat borderline and calls for additional data. We conclude that a linear relation is somewhat likely.

If we (possibly with additional data) determine that a linear function is the correct model, then the unit differences give the annual increases. As the differences are not all equal, we should use the average as our prediction for the yearly increase.

$$\frac{373.4 + 367.2 + 317.7}{3} = \frac{1058.3}{3} \approx 352.8.$$

Thus, the linear model would predict (on average) a yearly increase of about \$352.8 billion in market value.

Let's summarize:

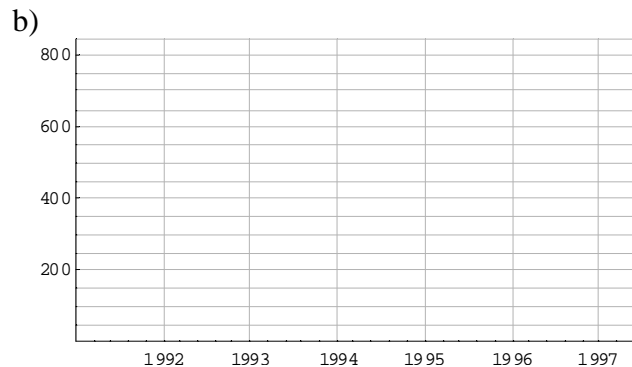
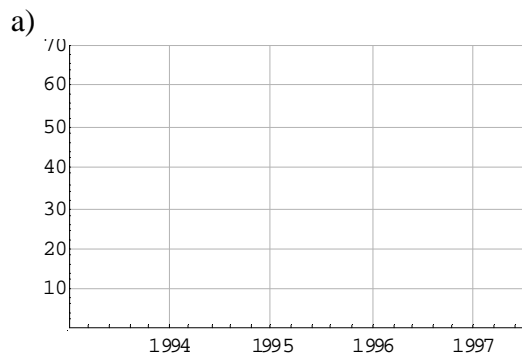
1. To check whether the given data follows a linear relation, we look first at the shape of the graph. If it looks somewhat like a straight line, we compute the first unit differences.
2. Select two consecutive input-output pairs, say  $(x_1, f(x_1))$  and  $(x_2, f(x_2))$  where  $x_1 < x_2$ . The first unit difference is computed as

$$\frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \left( \text{or } \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right).$$

3. If the differences for all the pairs in the table have about the same value, we conclude that a linear relationship is very likely. In this case, the average of the first unit differences gives the increase in output value per input unit.

#### Activity 3.1.4

For the two sets of data given below, follow the steps outlined above to determine whether a linear function (=model) is likely and give reasons why or why not. If a linear model is likely, determine the yearly increase in the market value and the value of the unexercised options, respectively.



a) Market value in \$ billion for Small Cap stocks <sup>2</sup>		
Year	Market Value	$\Delta$
1994	23.6	
1995	32.5	
1996	46.0	
1997	63.6	

b) Value of unexercised options at year end of Henry Silverman (CEO of Cendant) <sup>3</sup>		
Year	Value in \$ million	$\Delta$
1992	11	
1993	56	
1994	58	
1995	330	
1996	554	
1997	833	

<sup>2</sup> <http://www.nad.com/mr3b.html> (NASDAQ web site)

<sup>3</sup> Wall Street Journal, June 17, 1998