6.6 Equilibrium Values and their Stability for First-Order Non-Linear DDS

In Section 6.5 we have derived a formula to determine the equilibrium value for a first-order linear DDS and also a criterion to determine the stability of the equilibrium value. Unfortunately, we will not be able to derive such a general formula for a first-order non-linear DDS. However, we can use the same approach for finding the equilibrium as we did in the case of a linear DDS. Recall that for a system in equilibrium, the new output value $x(n)$ is equal to the old output value $x(n\text{-}1)$, i.e., no change occurs. Thus, if the system is given in the form

$$x(n) = f(x(n-1))$$

then this requirement for the equilibrium becomes

$$x(n) = f(x(n-1)) = f(x(n)).$$

If we again denote the equilibrium value by x, we need to solve the following equation:

$$x = f(x).$$

A value that solves this equation is called a *fixed point* of the function f. We summarize this in the following theorem.

Theorem 5 (Equilibrium Value for Non-Linear First-Order DDS)
The equilibrium values of a general first-order DDS of the form

$$x(n) = f(x(n-1))$$

are the fixed points of the function f. The equilibrium values can be computed by solving the equation

$$x = f(x)$$

for x.

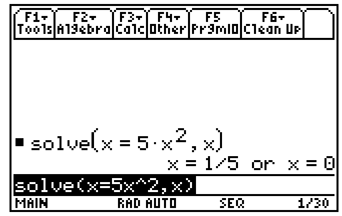
<u>Example</u>:
We want to find the equilibrium value(s) for the DDS given by $x(n) = 5x(n-1)^2$. The model function f (= right hand side of the iterative model equation) is given by $f(x(n-1)) = 5x(n-1)^2$. To find the equilibrium, we have to solve the following equation:

$$x = f(x) \quad \text{or} \quad x = 5x^2.$$

To find the solution(s), we can use either the TI-89 function **solve** or analytical methods (depending on your mathematics background). To use **solve**, we must recall that the first argument is the equation, and the second argument is the variable to be solved for.

Press [HOME] to return to the Home screen. Press [CATALOG] [S] and move down until the function **solve** is highlighted. Press [ENTER], then type

[X] [=] 5 [X] [^] 2 [,] [X] [)] [ENTER].



The answer indicates that both $x = \frac{1}{5}$ and $x = 0$ are equilibrium values. We can verify this by substituting these values into the original equation.

To solve the above equation analytically, we use methods to solve quadratic equations. We start by moving all terms to one side of the equation. The next step is to look whether factoring is possible. If not, apply the quadratic formula (see appendix, A4).

$$\begin{aligned} & x = 5x^2 \\ \Rightarrow \quad & 0 = 5x^2 - x \\ \Rightarrow \quad & 0 = x(5x-1) \end{aligned}$$

If a product is zero, then either one or both of the factors has to be zero. This implies that either

$$\begin{aligned} & x = 0 \quad \text{or} \quad 5x - 1 = 0 \\ \Rightarrow \quad & x = 0 \quad \text{or} \quad 5x = 1 \\ \Rightarrow \quad & x = 0 \quad \text{or} \quad x = \tfrac{1}{5} \end{aligned}$$

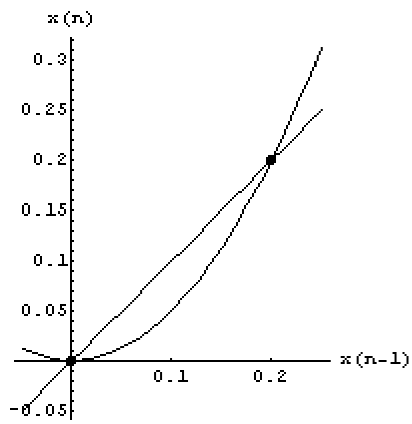
Again, both $x = 0$ and $x = \frac{1}{5}$ are the equilibrium values.

Activity 6.6.1

For the DDS given below, find the equilibrium value(s) by using either the TI-89 function **solve** or appropriate analytical methods.

a) $x(n) = x(n-1)\cdot(1 - x(n-1))$
b) $x(n) = x(n-1) - 2x(n-1)^2 + 3x(n-1)^3$
c) $x(n) = x(n-1)^2 + x(n-1)^3$

Graphically, we can find fixed points of the function *f* by drawing the graph of the function together with the line $y = x$, the 45° line. The *x*-values of the points at which the graph intersects the 45° line are the fixed points or equilibrium values. Why is this the case? Remember that in this graph, the input value is given by $x(n\text{-}1)$ (on the horizontal axis), and the output value is given by $f(x(n\text{-}1))$ (on the vertical axis). At the 45° line, the two values are identical, hence $x(n-1) = f(x(n-1)) = x(n)$. Thus, the system is in equilibrium for this input value. We will illustrate this method for the previous example, $x(n) = 5x(n-1)^2$. We graph the function $f(x) = x^2$ together with the 45° line.



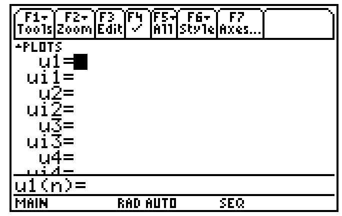
From the graph, we identify the points of intersection, (0, 0) and (0.2, 0.2), respectively. Their input values, namely 0 and 0.2 = 1/5, are the two equilibrium values.

We now turn to determining the stability of the equilibrium values for a non-linear DDS. The criteria are still the same:

- If the initial value is slightly above or below the equilibrium value and the sequence of system values converges (= gets close) to that equilibrium value, then the equilibrium is **stable**.
- If the initial value is slightly above or below the equilibrium value and the sequence of system values diverges away from that equilibrium value, then the equilibrium is **unstable**.
- If the sequences of system values for initial value slightly above or below the equilibrium value exhibit both converging and diverging behavior, then the equilibrium is **semi-stable**.
- In all other cases, the equilibrium is **neutral**.

To determine stability, we can look at a table of values or a web plot (also called Cob-web diagram). Let's start with the table, which will also help us determine the parameters for the graphing window of the web plot. In the previous example, $x(n) = 5x(n-1)^2$, we found that the equilibrium values are $x = 0$ and $x = 0.2$. We first check the equilibrium value $x = 0$ for stability. We need to look at the behavior of sequences that have initial values close to 0; for example, we can use 0.1 and -0.1. (Be careful not to chose a value that is bigger than the second equilibrium value 0.2.)

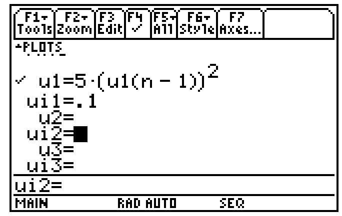
1. Press [◆][Y=] and press [F1] **8** to clear any previously defined sequences. Press [ENTER] to confirm your desire to start out with a clean screen. (If your screen does not display sequences u1, u2,…, press [MODE] and select SEQUENCE→ as the Graph mode.)



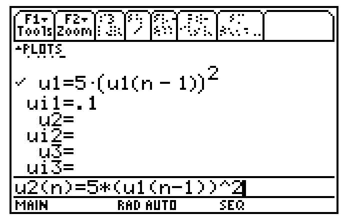
2. Move the cursor to the u1= position and type

   5 [×][(] [alpha][U] 1 [(] [alpha][N] [−] 1[)] [)] [^] 2 [ENTER].

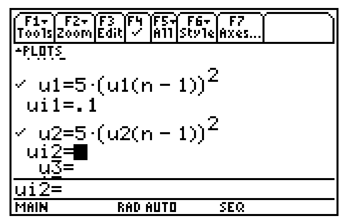
   As the initial value, type 0.1 and press [ENTER].



3. The second sequence is the same, it just has a different starting value. We can avoid retyping the recursive equation by using the copy and paste features of the TI-89. Highlight the expression for u1. Now press [◆][COPY]. Move the cursor to the u2= position and press [◆][PASTE]. This will paste the copied expression onto the entry line, where it can now be modified.

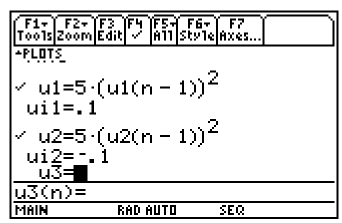


4. Even though it is the same sequence, we need to make one modification, namely changing u1 into u2. Move to the right of u1 and delete the number 1 with the [←] key. Type a 2 and press [ENTER].

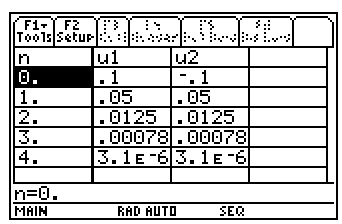


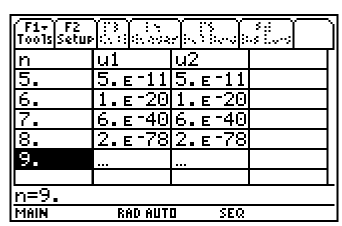
5. Enter the second initial value, -0.1, by typing

   [(-)] 0 [.] 1 [ENTER].



6. To see both sequences displayed together in a table, press [◆][TblSet] and set **tblStart** to **0** and **Δtbl** to **1**. Make sure that **Independent** is set to AUTO→. Press [ENTER], then [◆][TABLE]. To see additional values for larger *n*, use the [2nd]⊖ key to move down a screen at a time.





From the table, we see that the value for n = 4 is given as 3.1 E⁻6 (for both sequences). This format is called scientific notation and is a shorthand for $3.1 \cdot 10^{-6} = 0.0000031$. (The exponent -6 indicates that the decimal point is to be moved six places to the left, and the extra positions are to be filled with zeros.) The value for n = 4 is already quite close to zero. Values for n > 5 have exponents that are even larger in magnitude, which makes the resulting values even closer to 0. Therefore, sequences with initial values slightly above or below the equilibrium value, 0, approach the equilibrium value. Thus this equilibrium value is **stable**.

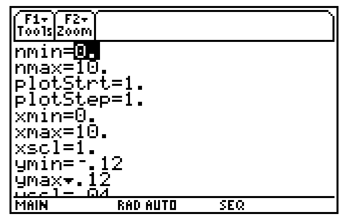
We can visualize this behavior by graphing the two sequences together.

1. Press [◆][Y=], then [2nd][F7]. Check that **Axes** is set to TIME→ and press [ENTER]. Press [◆][WINDOW] and set the parameter values as follows:

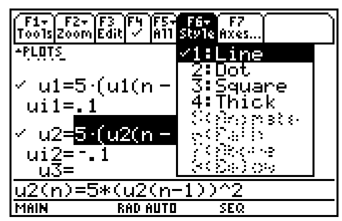
   nmin = 0 nmax = 10 plotStrt = 1 plotStep = 1

   xmin = 0 xmax = 10 xscl = 1

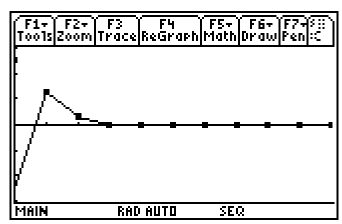
   ymin = -0.12 ymax= 0.12 yscl = 0.04



2. Press [ENTER], followed by [◆][Y=]. We will draw the two sequences in different styles. Highlight the second sequence, u2, and press [2nd][F6] and select **1:Line**.

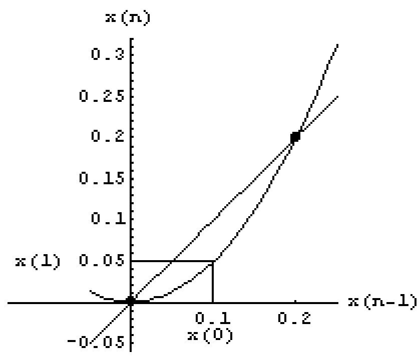


3. Press [ENTER], then [◆][GRAPH] to see the two sequences graphed together. The first sequence is displayed with squares, whereas the second one is graphed by connecting the values with a line. Notice that the two sequences coincide starting with n = 1.

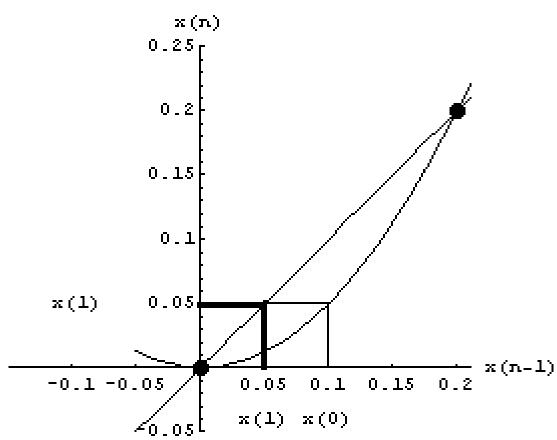


Now we will look at how a *Cob-web diagram* or *Cob-web graph* is constructed. This type of graph is an alternative way to visualize the sequence behavior over time. Unlike the graphs we have seen so far, which have time as the independent variable (and thus "represent" the explicit solution), a Cob-web graph "represents" the recursive model equation. In this type of graph, the old output value $x(n - 1)$ is displayed on the horizontal axis, and the new output value $x(n)$ is displayed on the vertical axis.

We start by graphing the model function f together with the 45° line (where $x(n) = x(n\text{-}1)$). The horizontal axis displays the values for $x(n-1)$ (=input values) and we can read off the output values $x(n)$ as $f(x(n-1))$. Starting with an input value, we will use horizontal and vertical lines to read off the corresponding output value and to translate this value from the vertical axis to the horizontal axis (so it becomes the next input value). First, we draw a vertical line through the initial value $x(0)$ until this line intersects with the graph of the model function f. The corresponding output value can be read off by drawing a horizontal line from the intersection point to the vertical axis. This gives the value of $x(1)$. Below is an illustration of the procedure for the iterative model equation $x(n) = 5x(n-1)^2$ with initial value $x(0) = 0.1$. In this case, $x(1) = 0.05$.

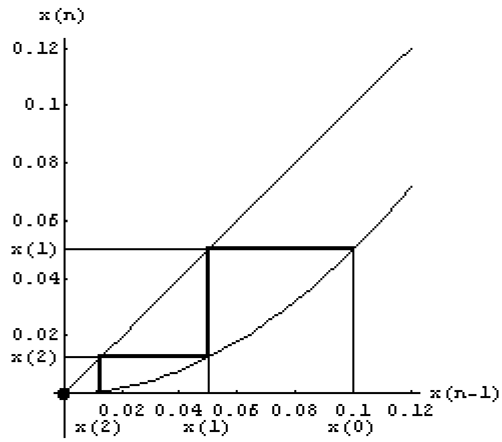


In order to use $x(1)$ as the next input value, we have to transfer its value from the vertical to the horizontal axis. We could just read it off, and then mark the appropriate value on the horizontal axis. Alternatively, we can use the 45° line (where input and output values are identical) to assist with the transfer. To make the output value into an input value, we draw a horizontal line from $x(1)$ to the 45° line, then a vertical line down to the horizontal axis (thick lines in the graph below). This transforms the value of $x(1)$ from output to input value.



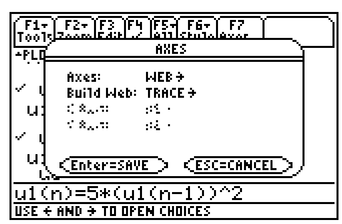
We repeat this process, now starting with $x(1)$, until we have transferred the value of $x(2)$ to the horizontal axis. You may notice that each step creates a number of lines, parts of which are drawn twice. These segments, as well as the initial vertical line, are shown as thin lines in the next graph.

All other segments are drawn as thick lines; they constitute the Cob-web diagram and are the only ones usually drawn.



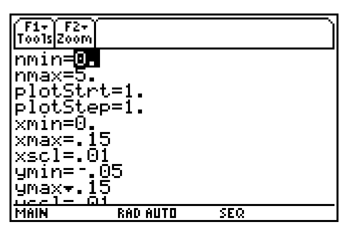
The TI-89 can display sequences as a web plot (= Cob-web diagram). This web plot can either be created step-by-step or displayed as the finished graph which then can be traced. We will take the example above and see how we can create the web plot for the recursive model equation $x(n) = 5x(n-1)^2$ with initial value $x(0) = 0.1$. In the following, references to *x*(1), etc. refer to the graphs above. Remember that *x*(1) correspond to u1(1), etc.

1. Press [◆][Y=] to return to the screen where the sequences are defined. Highlight the first sequence, press [2nd][F7] and set **Axes** to WEB→ and **Build Web** to TRACE→. This last setting creates the web plot one step at a time.



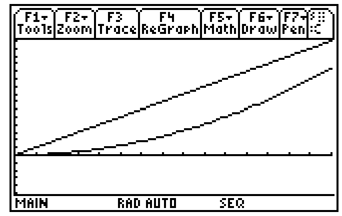
2. Press [ENTER], then [◆][WINDOW] and set the parameters as follows:

nmin = 0	nmax = 5	plotStrt = 1	plotStep = 1
xmin = 0	xmax = 0.15	xscl = 0.01	
ymin = -0.05	ymax= 0.15	yscl = 0.01	

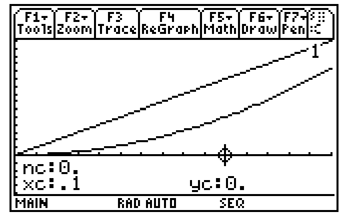


Note: Setting ymin to a value that is smaller than the minimal output value creates an area not covered by the web plot for easy read-off of the traced values.

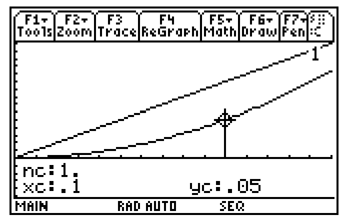
3. Press [ENTER], then [◆][GRAPH]. The line $y = x$ and the model function $f(x) = 5x^2$ are graphed together. Note that the line is not at 45 degrees, even though the *x*- and *y*-scales are the same. This is due to the fact that the graphing window is rectangular, not squared.



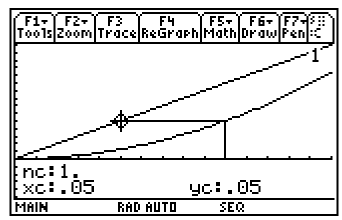
4. To see the step-by-step creation of the web plot, press [F3]. This activates the trace and puts the trace cursor at the initial value on the horizontal axis. Note that three values are displayed: **nc**, **xc**, and **yc**. Initially, **nc = nmin**, **xc = initial value for sequence**, and **yc = 0.** The number 1 in the upper right corner indicates that this plot is for the sequence u1.



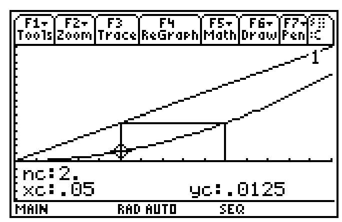
5. Press the ▶ key. The trace has moved up to the model function, which will compute *x*(1). Notice the updated values on the screen: **nc = 1**, **xc = 0.1** (not changed), and **yc = 0.05 = u1(xc) = u1(0.1)**. (This is the value of *x*(1).)



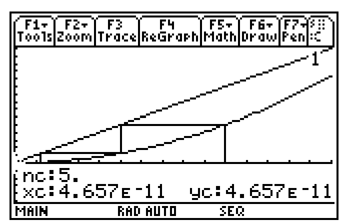
6. Press the ▶ key again. Now a horizontal line to the $y = x$ line has been drawn, i.e., the transfer from output to input value has taken place. Notice that **xc = yc**, and that **nc** remains the same. One cycle of computation has been completed.



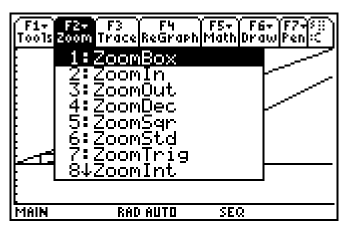
7. Press the ▶ key. This computes *x*(2) as indicated by the values on the screen: **nc = 2**, **xc = .05** (= *x*(1)), and **yc = .0125** (= *x*(2)).



8. Press the ▶ key repeatedly. The trace will stop when **nc = 5** (the value chosen for **nmax**). You can "undo" the web plot by using the ◀ key. (Just as in a regular graph, tracing is done only with the ▶ and ◀ keys. The ▶ key moves to larger (*n*-) values, and the ◀ key moves to smaller (*n*-) values.)



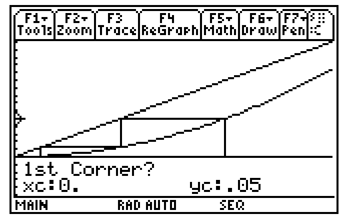
9. Let's zoom in at the current location of the trace cursor to see more details. This can be achieved by pressing [F2] and selecting an appropriate zoom option. We will use the option **1:ZoomBox**, which lets you draw a box to zoom in on.



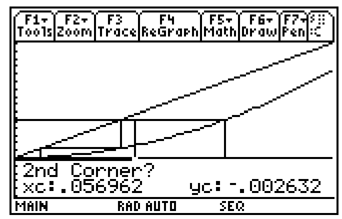
TIP:

When selecting any zoom option, keep in mind that it will change the values of the window parameters. The changes can be made permanent ([F2] [alpha][B] **2**) or you can return to the previous screen ([F2] [alpha][B] **1**). For more details on zooming, see Calculator Lesson T6.

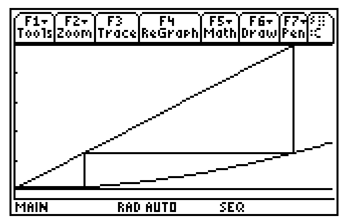
10. Press [ENTER]. You will be queried for the 1st corner, which can be any corner of the box you want to define. Use the ⬆ key to move to a location with coordinates **xc = 0** and **yc ≈ 0.05**.



11. Press [ENTER]. You will be asked for the second corner. Use the appropriate arrow keys to move to the opposite corner of the box, with coordinates **xc ≈ 0.6** and **yc ≈ 0**. As you move the cursor, the corresponding box is drawn.

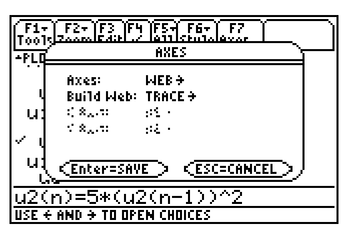


12. Press [ENTER]. The graph of the *y* = *x* line and the model function will be redrawn. To see the web plot, press [F3] and use the ➡ key repeatedly. Alternatively, press [◆][Y=], and with u1 highlighted, press [2nd] [F7]. Set **Build Web** to AUTO→ and then press [ENTER]. Press [◆] [GRAPH] to see the web plot drawn for *n* = nmin to *n* = nmax. Notice that in this mode, you need to activate the trace feature to see the values **nc**, **xc**, and **yc**.



In the enlarged portion of the graph we can see that the zig-zag line moves toward the equilibrium point (0,0) (input value = equilibrium value = 0). This shows that if we start with an initial value slightly above the equilibrium value, then the sequence of values converges towards the equilibrium value. Next we create a web plot for an initial value slightly below the equilibrium value, e. g. *x*(0) = -0.1.

1. Press [◆][Y=] and deselect u1 using the [F4] key. Check that u2 is selected (if not, highlight the sequence u2, which has the desired initial value of -0.1, and press [F4]). Press [2nd][F7]and set **Build Web** to TRACE→.



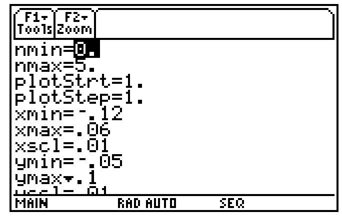
2. Press [ENTER]. Press [◆][WINDOW] to set appropriate parameters for this sequence (refer to the table of values computed earlier). Good values are:

   nmin = 0 nmax = 5 plotStrt = 1 plotStep = 1

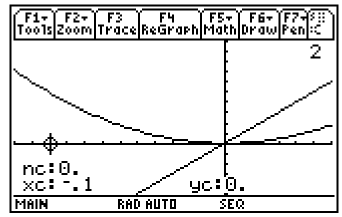
   xmin = -0.12 xmax = 0.06 xscl = 0.01

   ymin = -0.05 ymax= 0.1 yscl = 0.01

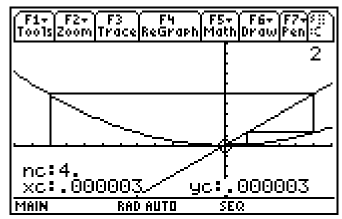
   (Again we make ymin a bit smaller than necessary to create space for the display of the trace values.)



3. Press [ENTER], then [◆][GRAPH]. Press [F3] to activate the trace. Note that now the number 2 is displayed at the upper right corner of the screen, indicating that this is a web plot of the sequence u2.



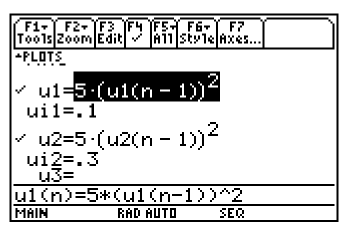
4. Press the ⊙ key repeatedly. Again we see that the sequence moves toward the point with coordinates **xc** = 0, the equilibrium point.



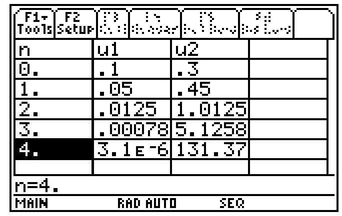
Since the sequence of values approaches the equilibrium value $x = 0$, whether $x(0)$ is slightly above or below this equilibrium value, we conclude that $x = 0$ is a **stable** equilibrium.

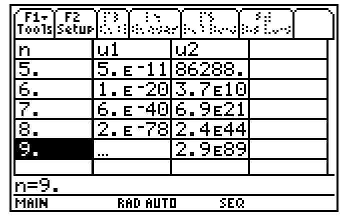
Let's now check the second equilibrium $x = \frac{1}{5} = 0.2$ using a table of values and a web plot, with initial values $x(0) = 0.1$ and $x(0) = 0.3$. We have already seen the web plot for $x(0) = 0.1$ and know that the values tend toward 0, not toward the equilibrium value $x = 0.2$. This indicates that this equilibrium cannot be stable, but must be either unstable or semi-stable. Let's see what happens for the initial value $x(0) = 0.3$.

1. As we want to see what happens for initial values 0.1 and 0.3, we can just change one of the previously defined initial values. Press [◆][Y=] and highlight the initial value for u2. Press [ENTER], which will move the cursor to the entry line. Type 0.3 and press [ENTER]. Select both u1 and u2 (if not already selected) using the [F4] key.



2. To see both sequences displayed together in a table, press [◆][TblSet] and set **tblStart** to **0** and **Δtbl** to **1**. Make sure that **Independent** is set to AUTO→. Press [ENTER], then [◆][TABLE]. To see additional values for larger *n*, use the [2nd]⊝ key to move down a screen at a time.





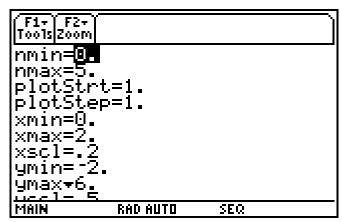
Notice that the values in the sequence with initial value 0.3 become very large, very rapidly. The values are again displayed in scientific notation, except this time the exponents are positive. For example, 3.7 E10 is shorthand for $3.7 \cdot 10^{10} = 37000000000$ (the exponent (+)10 indicates that the decimal point is to be moved ten places to the right, and the extra positions are to be filled with zeros). Thus, the sequence of values does not move closer and closer to x = 0.2, but rather moves rapidly away from it. Let's look at the corresponding web plot.

3. Press [◆][WINDOW] to set appropriate parameters for this sequence (refer to the table of values computed in Step 2). Good values are:

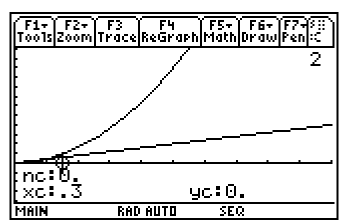
   nmin = 0 nmax = 5 plotStrt = 1 plotStep = 1

   xmin = 0 xmax = 2 xscl = 0.2

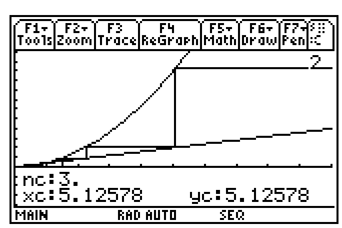
   ymin = -2 ymax = 6 yscl = 0.5



4. Press [ENTER], then [◆][GRAPH]. Once the BUSY sign in the lower right corner has disappeared, press [F3] to activate the trace. If the number 1 is displayed in the upper right corner of the screen, use the ⊝ key to switch to the next sequence (only one sequence can be plotted at a time in a web plot).



5. Press the ⊳ key repeatedly. Now the trace line moves away from the equilibrium point (= intersection of the model function with the line *y* = *x*), a sign that this equilibrium value cannot be stable.



Since the sequence of values move away from the equilibrium value *x* = 0.2, both when the initial values are below and above the equilibrium value, $x = \frac{1}{5} = 0.2$ is an **unstable** equilibrium.

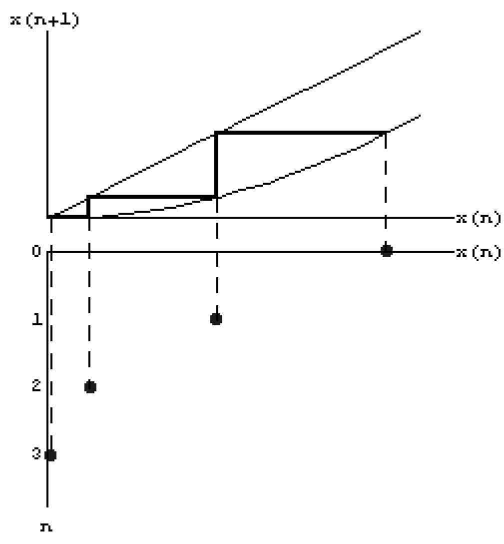
Activity 6.6.2

For the DDS given below, determine the stability of the equilibrium value(s) in three different ways, by using 1) the table of values, 2) a graph of the sequence where **Axes** is set to TIME→, and 3) a web plot of the sequence. (Note: These are the DDS from Activity 6.6.1)

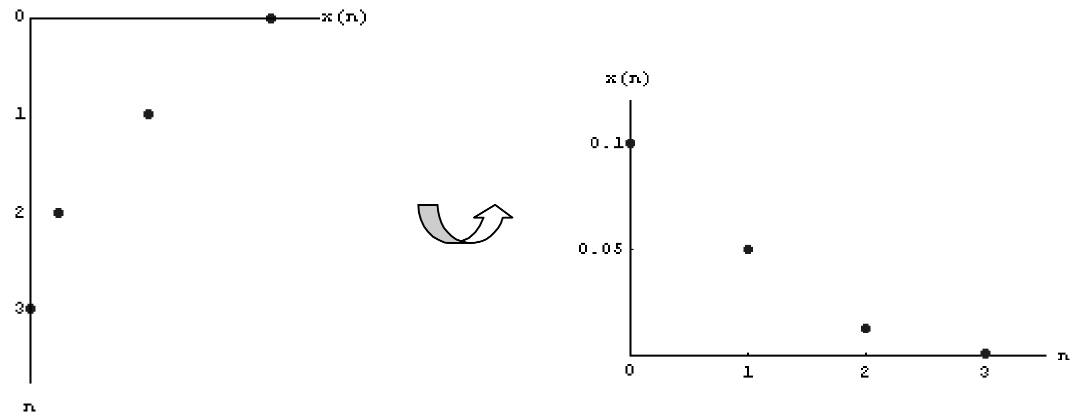
a) $x(n) = x(n-1)\cdot(1-x(n-1))$
b) $x(n) = x(n-1) - 2x(n-1)^2 + 3x(n-1)^3$
c) $x(n) = x(n-1)^2 + x(n-1)^3$

To conclude this section, we will have a closer look at the two types of graphs that we have used to determine stability visually. Recall that the web plot represents the recursive model equation (where $x(n)$ is a function of $x(n\text{-}1)$), whereas the other type of graph represents the explicit model solution (where $x(n)$ is a function of n). As they both show the sequence of output values, there should be a way of "connecting" them.

Below is an illustration of how the two graphs relate to each other. We start by creating a Cob-web diagram. Below this graph we put another set of horizontal and vertical axes. On the horizontal axis we mark $x(n)$, on the vertical axis we mark the time n. Now we extend the vertical lines of the Cob-web graph downwards (through the value $x(k)$) until we reach time k and plot a point at this position.



If we now take the lower part of this graph and rotate it by 90°, then we get a graph corresponding to the explicit solution (created when the **Axes** are set to TIME→ rather than WEB→).


0
1
2
3
x(n)
n
x(n)
0.1
0.05
0
1
2
3
n