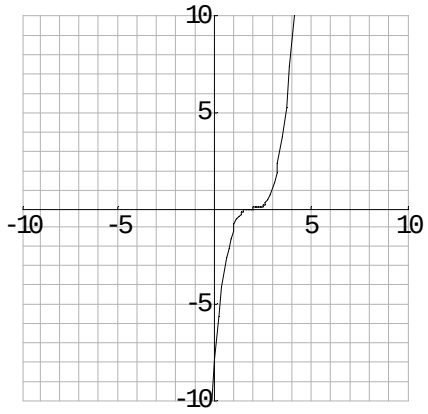
3.3 Cubic Functions

A *cubic function* is described by

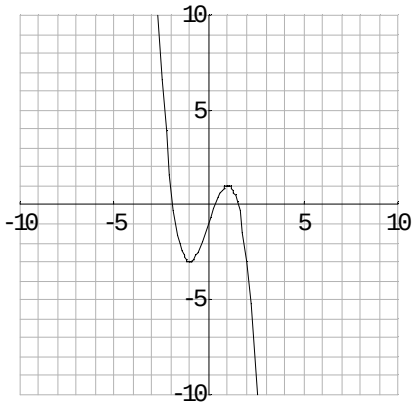
$$f(x) = ax^3 + bx^2 + cx + d\,,$$

where a, b, c, and d are constants and $a \neq 0$. The highest power for the input variable is 3, hence the name cubic. Graphs of cubic functions show a bit more variety than those for linear or quadratic functions. Here are some examples of cubic functions with their graphs:

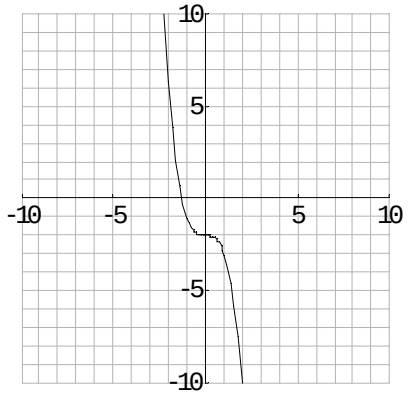
a) $f(x) = x^3 - 6x^2 + 12x - 8$



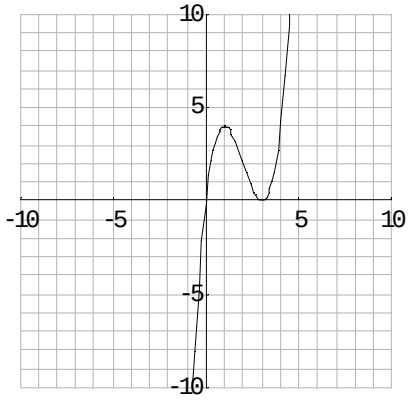
b) $g(t) = -t^3 + 3t - 1$



c) $h(u) = -u^3 - 2$



d) $f(t) = t^3 - 6t^2 + 9t$



Activity 3.3.1

For the cubic functions given above, read off the value of the constants *a*, *b*, *c*, and *d*. Make a conjecture about what influence the sign of the constant *a* has on the shape of the graph.

The common feature of these graphs is that there is exactly **one** change in curvature (from hill to bowl or vice versa). Furthermore, as the input values become either very small or very large, the output values grow or decrease without bound, with opposite behaviors at the two "ends" of the graph.

A cubic function is either always increasing, always decreasing or switches:

increasing $\rightarrow$ decreasing $\rightarrow$ increasing ("mostly increasing") or

decreasing $\rightarrow$ increasing $\rightarrow$ decreasing ("mostly decreasing")

Finally, cubic functions have constant third unit differences, defined analogously to the second unit differences. However, we rarely use this numerical tool to check if the data comes from an underlying cubic function. The shape of the graph is distinct from the graphs of the other types, and thus suffices for an initial determination.

Activity 3.3.2

For each of the four examples of cubic functions given previously, decide whether the function is increasing, decreasing, "mostly increasing," or "mostly decreasing."

a) $f(x) = x^3 - 6x^2 + 12x - 8$ b) $g(t) = -t^3 + 3t - 1$

c) $h(u) = -u^3 - 2$ d) $f(t) = t^3 - 6t^2 + 9t$

Can you guess what influence the sign of the constant a has on the shape of the graph?

Here is a summary of the properties of a cubic function.

Properties of Cubic Functions

1) The functional expression of a cubic function is given by $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$.
2) A cubic function always increases, always decreases, or switches: increasing-decreasing-increasing ("mostly increasing") or decreasing-increasing-decreasing ("mostly decreasing").
3) The graph of a cubic function changes curvature once, from hill to bowl or vice versa. On one side of the graph, the output values increase without bound and on the other side the output values decrease without bound.

Activity 3.3.3

Draw a graph for the sets of data as reported in the USDA Statistical Highlights. From the graph, determine whether or not a cubic function may have produced the data. Give reasons for your answers.

a) Number of acres of fresh carrots harvested in the United States[1]

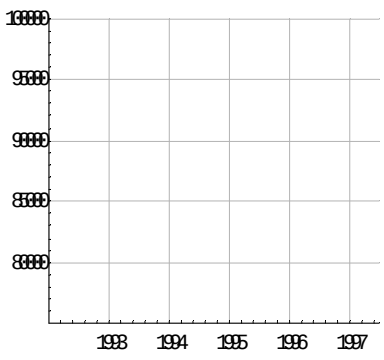
Year	Acres
1993	78,220
1994	74,630
1995	81,120
1996	92,160
1997	97,460

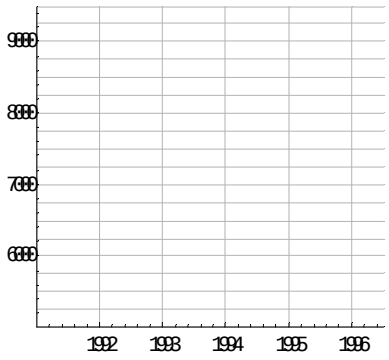
b) Number of bales of cotton (in thousands) exported[2]

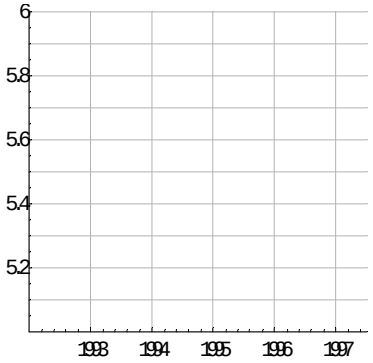
Year	Acres
1992	5,200
1993	6,860
1994	9,400
1995	7,680
1996	6,870

c) Yield per acre (in thousands) for processed cucumbers[1]

Year	Tons (in 1000s)
1993	5.38
1994	5.44
1995	5.22
1996	5.37
1997	5.98







Remark: Linear, quadratic, and cubic functions are special cases of *polynomials*. A *polynomial of degree n* has the form

$$p_n(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

where $c_0, c_1, \ldots, c_{n-1}$, and c_n are constants with $c_n \neq 0$. The degree of the polynomial indicates the highest power of the independent variable. Therefore, a linear function is a polynomial of degree 1, a quadratic function is a polynomial of degree 2, and finally a cubic function is a polynomial of degree 3.

[1] www.usda.gov/nass/stathigh/1998/cr-vg-ap.htm

[2] www.usda.gov/nass/pubs/stathigh/1998/ec-ex.htm