

### 3.2 Quadratic Functions

A *quadratic function* is described in one of two alternative forms:

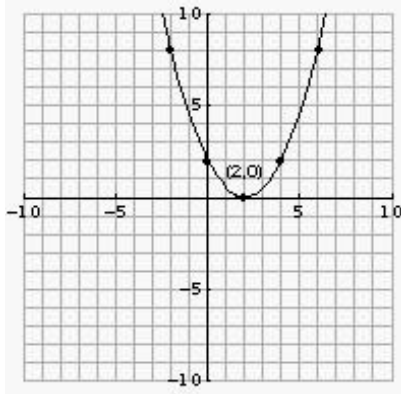
$$f(x) = ax^2 + bx + c \quad (1) \quad \text{or} \quad f(x) = a(x-h)^2 + k \quad (2)$$

where  $a$ ,  $b$ ,  $c$ ,  $h$  and  $k$  are constants and  $a \neq 0$  (otherwise we would have a linear function). Note that the highest power for the independent variable is 2 (unlike for linear functions where the highest power for the independent variable is 1). The graph of a quadratic function, called a *parabola*, has the shape of a hill or a bowl. It has either a high or a low point, called the *vertex*, and does not change curvature. Furthermore, the graph is symmetrical to a vertical line through the vertex, i.e., if a mirror is placed on this line, then the reflection of the graph and the graph itself coincide.

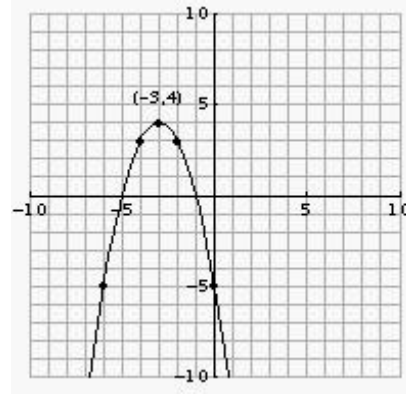
#### Activity 3.2.1

For each function given below with its graph, identify the value of the constants  $a$ ,  $b$ , and  $c$  (if the function is given in form (1)) or the value of the constants  $a$ ,  $h$ , and  $k$  (if given in form (2)).

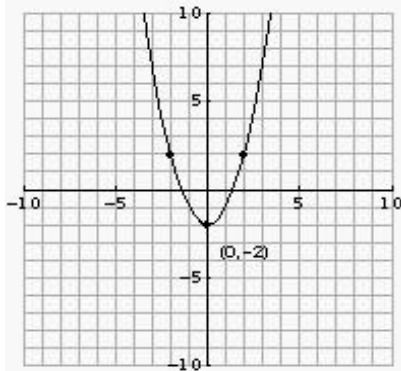
a)  $f(x) = \frac{1}{2}(x-2)^2$



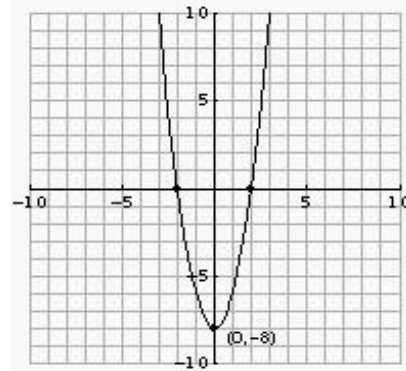
b)  $g(t) = -t^2 - 6t - 5$



c)  $f(u) = u^2 - 2$



d)  $f(t) = 2t^2 - 8$



a)	$f(x) = \frac{1}{2}(x-2)^2$	$a =$	$h =$	$k =$
b)	$g(t) = -t^2 - 6t - 5$	$a =$	$b =$	$c =$
c)	$f(u) = u^2 - 2$	$a =$	$b =$	$c =$
d)	$f(t) = 2t^2 - 8$	$a =$	$b =$	$c =$

The two alternative expressions for a quadratic function are equivalent, in that we can transform a function given in either form to the other. To go from form (2) to form (1), we expand the first term, the binomial square. Here is an example for the function  $f(x) = \frac{1}{2}(x-2)^2 + 3$ . The expression is in form (2) with  $a = \frac{1}{2}$ ,  $h = 2$  and  $k = 3$ . Now let's do the math.

$$\begin{aligned}
 f(x) &= \frac{1}{2}(x-2)^2 + 3 \\
 &= \frac{1}{2}(x-2)(x-2) + 3 \\
 &= \frac{1}{2}(x^2 - 2x - 2x + 4) + 3 \\
 &= \frac{1}{2}(x^2 - 4x + 4) + 3 \\
 &= \frac{1}{2}x^2 - 2x + 2 + 3 \\
 &= \frac{1}{2}x^2 - 2x + 5
 \end{aligned}$$

Now the function is expressed in form (1) with  $a = \frac{1}{2}$ ,  $b = -2$ , and  $c = 5$ . For the reverse operation, to go from form (1) to form (2), we need to employ the method of **completing the square**. This procedure reverses expansion, i.e., it tries to condense the expanded terms. Several examples and a detailed step-by-step description of this method are given in appendix A2.

Why are there two different formulas for quadratic functions? Each of the two forms serves a specific purpose: Form (1) shows that a quadratic function is a polynomial of degree 2, a generalization of a linear function (a square term is added). This form clearly shows the highest degree for  $x$ , namely 2. Form (2), on the other hand, tells us more about the geometric features of the quadratic function. From the expression  $f(x) = a(x-h)^2 + k$  we can easily read off the coordinates of the vertex,  $(h, k)$ .

How can we read off the coordinates of the vertex if a function is given in form (1)? We will need to transform it into form (2). To save ourselves the work of performing this procedure every time we face this question, the procedure of completing the square can be performed for the function  $f(x) = ax^2 + bx + c$  for general constants rather than with specific values for  $a$ ,  $b$ , and  $c$ . The resulting function in form (2) is given by

$$\begin{aligned}
 f(x) &= a \cdot \left( x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a} \\
 &= a \cdot \left( x - \underbrace{\left( -\frac{b}{2a} \right)}_h \right)^2 + \underbrace{\frac{4ac - b^2}{4a}}_k
 \end{aligned}$$

This might look intimidating, but we can read off the values for  $h$  and  $k$  should be:

$$h = -\frac{b}{2a} \text{ and } k = \frac{4ac - b^2}{4a}.$$

Now, to determine the vertex coordinates of a quadratic function given in form (1) we simply follow these steps:

- 1) Read off the values of  $a$ ,  $b$ , and  $c$ .
- 2) Substitute these values into the formulas for  $h$  and  $k$  given above.
- 3) The vertex is given by  $(h, k)$ .

Let's check this in our example,  $f(x) = \frac{1}{2}x^2 - 2x + 5$ . We start by reading off the values of the constants  $a = \frac{1}{2}$ ,  $b = -2$ ,  $c = 5$ . We substitute these values:

$$\begin{aligned}
 h &= -\frac{b}{2a} = -\frac{(-2)}{2 \cdot \frac{1}{2}} = \frac{2}{1} = 2 \\
 k &= \frac{4ac - b^2}{4a} = \frac{4\left(\frac{1}{2}\right)5 - (-2)^2}{4\left(\frac{1}{2}\right)} = \frac{10 - 4}{2} = \frac{6}{2} = 3.
 \end{aligned}$$

Thus, the vertex is at (2,3), which checks with form (2) of our function:

$$f(x) = \frac{1}{2}(x - 2)^2 + 3$$

In summary, we have the following formulas for the vertex:

Functional Form	Vertex Formula
$f(x) = ax^2 + bx + c$	$\left( -\frac{b}{2a}, \frac{4ac - b^2}{4a} \right)$
$f(x) = a(x - h)^2 + k$	$(h, k)$

**Activity 3.2.2**

- 1) For the functions given in Activity 3.2.1, use the appropriate vertex formula to compute the vertex location. Then check the graph to see if the vertex is located at the point you found.
- 2) In both forms, the value of the constant  $a$  is the same. The sign of  $a$  (namely whether  $a$  is positive or negative) determines whether the parabola is hill-shaped or bowl-shaped. Use the graphs in Activity 3.2.1 to complete the rule below:

Rule: The graph is hill-shaped if  $a$  is \_\_\_\_\_.

The graph is bowl-shaped if  $a$  is \_\_\_\_\_.

So far, we have only considered graphical features of quadratic functions. Like linear functions, there are also numerical properties that can be checked for quadratic functions. In particular, quadratic functions have constant *second unit differences*. Computing these second unit differences is very similar to computing first unit differences, except we have to apply the procedure twice.

**Procedure for Computing Second Unit Differences:**

Select three consecutive input-output pairs, say  $(x_1, f(x_1))$ ,  $(x_2, f(x_2))$  and  $(x_3, f(x_3))$ , where  $x_1 < x_2 < x_3$ . Compute the first unit differences

$$d_1 = \frac{\Delta f}{\Delta x} = \frac{f(x_1) - f(x_2)}{x_1 - x_2} \quad \text{and} \quad d_2 = \frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_3)}{x_2 - x_3}.$$

Now compute the second unit difference as  $\frac{\Delta d}{\Delta x} = \frac{d_1 - d_2}{x_1 - x_3}$ .

This procedure is illustrated in the example below:

**Example:**

Compute the second unit differences for the function  $f(x) = -3x^2 + 5$ . First, we create a table of input-output values. As usual, we select both negative and positive input values and put them in increasing order. Next, we select three consecutive pairs to compute one second unit difference.

Here's one computation in detail. We select the data points  $(-2,-7)$ ,  $(0,5)$  and  $(1,2)$  as  $(x_1, f(x_1))$ ,  $(x_2, f(x_2))$ , and  $(x_3, f(x_3))$ , respectively. Computing the first unit difference for the first and second points gives

$$d_1 = \frac{f(x_1) - f(x_2)}{x_1 - x_2} = \frac{-7 - 5}{-2 - 0} = \frac{-12}{-2} = 6.$$

For the second and third points, we get

$$d_2 = \frac{f(x_2) - f(x_3)}{x_2 - x_3} = \frac{5 - 2}{0 - 1} = \frac{3}{-1} = -3.$$

Now, we will use these two first unit differences,  $d_1$  and  $d_2$ , to compute the second unit difference. Note that in the denominator, we use the first and third  $x$ -values.

$$\text{second unit difference} = \frac{d_1 - d_2}{x_1 - x_3} = \frac{6 - (-3)}{-2 - 1} = \frac{9}{-3} = -3$$

Note that the  $x$ -value of the second data point does not get used in the last computation! A complete table of values with first and second unit differences is shown below. Compute the remaining two second unit differences to make sure you have understood the procedure.

$t$	$-3t^2 + 5$	First Unit Difference	Second Unit Difference
-3	-22	15	
-2	-7	6	$\frac{15-6}{-3-0} = \frac{9}{-3} = -3$
0	5	-3	$\frac{6-(-3)}{-2-1} = \frac{9}{-3} = -3$
1	2	-15	
4	-43	-30	
6	-103		

**Activity 3.2.3**

Make a table of values for the function  $h(u) = u^2 - 2$ . Select six pairs of input-output values and compute the corresponding second unit differences. (You should have five first unit differences and four second unit differences.)

$u$	$u^2 - 2$	First Unit Difference	Second Unit Difference

We can now summarize the properties of quadratic functions:

Properties of Quadratic Functions

- 1) The functional expression of a quadratic function can be written in two forms:  $f(x) = ax^2 + bx + c$  or  $f(x) = a(x - h)^2 + k$ , with  $a \neq 0$ .
- 2) A quadratic function has constant second unit differences.
- 3) The graph of a quadratic function is a parabola. It is hill or bowl-shaped depending on the sign of  $a$ . A parabola does not change curvature and is symmetrical with respect to a vertical line through the vertex.

Like the linear function, we have two types of properties:

- properties that can be verified numerically (property 2).
- properties that describes the shape of the graph of the data and can be seen in the graph itself (property 3).

Thus, our approach to check for a quadratic function is the same as for a linear function:

- 1) Graph the data.
- 2) Check whether the graph looks like a quadratic function, keeping in mind that we may see only part of the parabola, such as the increasing or decreasing half of the bowl or hill.
- 3) If a quadratic function seems likely, then compute the second unit differences from consecutive data points. If the resulting values are almost equal, a quadratic function is a good candidate.

### Activity 3.2.4

Graph the (partial) data on the median age of marriage given below. Does the graph suggest a quadratic function? Give a reason for your answer, keeping in mind that you may see only part of a parabola. If you think the shape indicates a quadratic function, compute the second unit differences to verify or reject your initial guess.

Year	Median Age	First Unit Diff.	Second Unit Diff.
1965	20.6		
1970	20.8		
1975	21.1		
1980	22.0		
1985	23.3		

