2. Functions and Their Representations

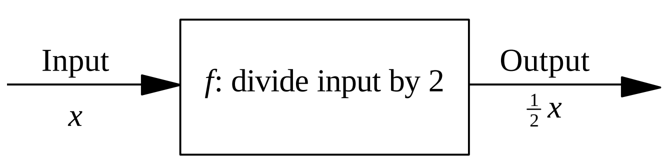
A *function* is a rule that assigns a **unique** output value $f(x)$ to a given input value x, that is, for every input value there is exactly one output value. Note that the name x for the input variable is a generic choice. Very often, the input variable is denoted by a different letter such as t. Likewise, the output variable is not always denoted by f. For example, if we are describing the size of a population in a specific year, then the input variable would be t = time in years, and the output variable would be $P(t)$ = population at time t. Thus, the naming of input and output variables reflects their meaning in a given context.

2.1 Functions Represented as Tables and Formulas

A function can be expressed in many different ways. Often it is given as a list of input-output pairs, especially when we are dealing with data. Here is an example:

input x	**output $f(x)$**
-1	-0.5
1	0.5
2	1
5	2.5
6	3

Tables are an easy way to describe a function, but they become tedious when many different input values are used. In this case, it would be beneficial if we could find a pattern that allows us to condense the whole table into a simple rule. Let's try that for the table of input-output pairs above. Do you see a pattern that we can write out as a rule? It looks as if the output is half of the input. Checking all the values we see that this pattern is valid for the given pairs. Therefore, we can say the *function rule* to get the output is "divide the input by 2." We can illustrate this as follows:



We can translate this diagram into a verbal description of the function:

output = half of input

Replacing *output* by $f(x)$ and *input* by x results in the *functional expression*

$$f(x) = \frac{1}{2}x$$

which condenses the table.

Activity 2.1.1

For the given tables of values, derive a function rule.

a)

x	$f(x)$
-2	4
-1	2
0	0
1	-2
2	-4
3	-6

b)

x	$f(x)$
-2	5
-1	4
0	3
1	2
2	1
3	0

c)

x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4
3	9

As in the previous example, first give a verbal description, then derive the functional expression.

a)

b)

c)

Very often, the input variable is called the *independent* variable, and the output variable is called the *dependent* variable. If we look at the diagram on the previous page, we can explain this naming convention. We are free (independent) to pick different input values, but for every input value, the output value is determined by the function rule. Thus, the output is **dependent** on the input, making it the dependent variable.

We have seen how to go from a table to a function rule. Sometimes this may be fairly obvious, sometimes it may be a bit harder or not even possible. However, going the other direction, creating a table from a function rule, always works. So let's try that using the example $f(x) = 2x + 3$. First, we select values for the input (= independent) variable. Usually, we start with simple values such as whole numbers. It is helpful if the values are written down in order, especially if you want to create a graph of the function later.

x	$f(x)$
-5	
-3	
-1	
0	
2	
4	

To compute the output values, we replace x in the function rule by the specific input value. For example, to compute the output value for $x = -5$, think of x as a place holder.

$$f(x) = 2x + 3$$
$$f(\_) = 2 \cdot \_ + 3$$

Since $x = -5$, replace every x by -5, i.e., put -5 in each place holder position.

$$\begin{aligned} f(-5) &= 2 \cdot (-5) + 3 \\ &= -10 + 3 \\ &= -7 \end{aligned}$$

(Read: The function (=output) value for -5 is -7.)

Let's try it again for a different value of x, say $x = 2$.

$$\begin{aligned} f(x) &= 2x + 3 \\ f(\_) &= 2 \cdot \_ + 3 \\ f(2) &= 2 \cdot 2 + 3 \\ &= 4 + 3 \\ &= 7 \end{aligned}$$

Our table now looks like this:

x	$f(x)$
-5	**-7**
-3	
-1	
0	
2	**7**
4	

Activity 2.1.2

Compute the remaining values for the table of the function $f(x) = 2x + 3$.

A function rule can have different descriptions for different input values. For example,

$$f(x) = \begin{cases} 1 & \text{if} \quad x < 0 \\ 2 & \text{if} \quad x = 0 \\ x^2 & \text{if} \quad x > 0 \end{cases}$$

Here is a table of some of the input-output pairs for this function:

x	$f(x)$
-2	1
-1	1
0	2
$\frac{1}{2}$	$\frac{1}{4}$
1	1
3	9

Let's see how the function values in the table above were computed. If $x = -2$, then it falls into the case $x < 0$. Therefore, $f(-2) = 1$. If $x = -1$, then it falls into the same case, $x < 0$. Hence, $f(-1) = 1$. If $x = 0$, then $f(0) = 2$. If $x = \frac{1}{2}$, then $x > 0$, and so $f(x) = x^2$. To compute the function value, we need to replace x by the value of $\frac{1}{2}$. Since $f(x) = x^2$, $f(\frac{1}{2}) = (\frac{1}{2})^2 = \frac{1}{4}$. If $x = 1$, then we are again in the case $x > 0$. Replacing x by 1 we get $f(1) = 1^2 = 1$. Finally, if $x = 3$, again the case $x > 0$ applies and we have $f(3) = 3^2 = 9$.

Therefore, if a function has a description that gives different rules for different values of the independent variable, you first need to check which case applies to the particular input value. Once the case is determined, just replace the input value in the corresponding functional expression for that case to compute the output value.

Activity 2.1.3

Compute the table of values for the functions given below.

a) $f(x) = -2x + 5$

x	$f(x)$
-4	
-2	
-1	
0	
1	
$\frac{3}{2}$	
3	

b) $f(x) = x^3$

x	$f(x)$
-2	
-1	
0	
$\frac{1}{2}$	
1	
3	

c) $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ -x & \text{if } x > 1 \end{cases}$

x	$f(x)$
-2	
-1	
0	
0.5	
1	
1.5	
2	
3	

The set of values for which the function can be computed is called the (*mathematical*) *domain*. Examples of values that are excluded from the domain are those that would result in division by zero or a negative value of which a square root is to be taken. For example, the function $f(t) = \frac{1}{t} + 3$ can be computed for all values of t except for $t = 0$, (as there would be division by zero). Thus, the mathematical domain consists of all numbers except 0.

If a function arises from the context of a problem, then a further restriction of possible input values may be necessary. For example, if the input variable t represents the age of a person, then negative values of t do not make sense, even if they are in the mathematical domain. We call the set of values from the domain that makes sense in the context of a problem the *context domain*.

Here is an example where the mathematical and context domain differ. Let r be the radius of a disk. Then, the area of the disk is given by $A(r) = p\, r^2$. In this case, the function $A(r)$ can be computed for all numbers, but in the context of the area of a disk, $A(r)$, makes sense only for positive numbers. Thus, the mathematical domain (all numbers) differs from the context domain (all positive numbers).

Activity 2.1.4

For each of the following functions, determine the mathematical domain. If a context is given, also determine the context domain.

a) $f(x) = x^2 - 2x$
b) $f(x) = \sqrt{x-2}$
c) $D(t) = t^2 - 2t$, where $D(t) =$ distance driven (in miles) and $t =$ time since beginning of trip (in hours)