A4 Solutions of a Quadratic Equation

To solve a *quadratic equation* of the form

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are constants with $a \neq 0$, we can use two different methods:

- factoring
- using the *quadratic formula*.

If the *quadratic form* $ax^2 + bx + c$ factors easily, i.e.,

$$ax^2 + bx + c = (x - d)(x - e)$$

for some constants *d* and *e*, then we can read off the solutions as $x_1 = d$ and $x_2 = e$. (These are the values subtracted from *x* in each factor.) We use the fact that if a product is zero, then at least one of the factors has to be zero:

$$\begin{aligned} & (x-d)(x-e) = 0 \\ \Leftrightarrow \quad & (x-d) = 0 \quad \text{or} \quad (x-e) = 0 \\ \Leftrightarrow \quad & x = d \quad \text{or} \quad x = e \end{aligned}$$

Factoring relies on the following four equalities:

$$\begin{aligned} (x+u)^2 &= x^2 + 2ux + u^2 \\ (x-u)^2 &= x^2 - 2ux + u^2 \\ (x+u)(x-u) &= x^2 - u^2 \\ (x+u)(x+v) &= x^2 + (u+v)x + uv \end{aligned}$$

Note that the last equation includes all others as a special case.

In the first two equations, the additive term is a perfect square, and the factor of *x* is twice (positive or negative) the number which was squared for the additive term. The third equation only applies if we have a difference of squares. The last equation uses the fact that the additive constant is a product of two numbers, and at the same time, the factor of *x* is the sum of those same numbers.

Example:

1) $x^2 - 1$

This quadratic form is a difference of two squares, therefore, the third equation applies and we can factor

$$x^2 - 1 = (x+1)(x-1)$$

Thus, the solutions are $x_1 = -1$ and $x_2 = 1$.

2) $x^2 - 2x - 3$

Since 3 is not a perfect square, we try to use the fourth equation. We are looking for two numbers whose product equals –3, and whose sum equals –2. A product of –3 can be created with integers only in two ways:

(-1)(3) = -3 and (-3)(1) = –3 .

The corresponding sums are

-1 + 3 = 2 and –3 + 1 = -2

Therefore, the values of *u* and *v* have to be –3 and 1, respectively, and the quadratic form factors as

$$x^2 - 2x - 3 = (x-3)(x+1)$$

Thus, the solutions of the quadratic equation are $x_1 = 3$ and $x_2 = -1$.

If the quadratic form does not have such a nice factorization, then we use the *quadratic formula*, which gives the following solutions to the quadratic equation:

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

i.e., $x_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

Note that we have to be careful about the square root; we can evaluate this term only if the radical $b^2 - 4ac$ is nonnegative. This leads to three different cases:

i) $b^2 - 4ac > 0$ (no problem with the radical $\rightarrow$ two different solutions)
ii) $b^2 - 4ac = 0$ (the radical is 0, therefore x_1 and x_2 are the same $\rightarrow$ one solution)
iii) $b^2 - 4ac < 0$ (the radical cannot be evaluated $\rightarrow$ no solution)

Example:

a) $2x^2 + 4x - 3 = 0$

Here a = 2, b = 4, c = -3. Thus,

$$x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 2 \cdot (-3)}}{2 \cdot 2} = \frac{-4 \pm \sqrt{16 + 24}}{4} = \frac{4 \pm \sqrt{40}}{4}$$

$\rightarrow$ two solutions: $x_1 = \dfrac{-4 + \sqrt{40}}{4}$ and $x_2 \dfrac{-4 - \sqrt{40}}{4}$.

(Note that the solutions can be simplified a bit more, but even that simplified answer will contain a square root).

b) $x^2 - 8x + 16 = 0$

Here a = 1, b = -8, c =16. Therefore,

$$x_{1,2} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \cdot 1 \cdot 16}}{2 \cdot 1} = \frac{8 \pm \sqrt{64 - 64}}{2} = \frac{8 \pm 0}{2} = 4$$

$\rightarrow$ one solution $x_1 = x_2 = 4$.

c) $x^2 + 5 = 0$

We read off a = 1, b = 0, c = 5. Thus,

$$x_{1,2} = \frac{-0 \pm \sqrt{0^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{\sqrt{-20}}{2} \rightarrow \text{ no solution.}$$

Remark: 1) The quadratic formula applies to every quadratic equation. If the quadratic form factors, then the solutions resulting from factoring and those resulting from the quadratic formula are identical.

2) Factoring is used when the constants in the quadratic form are relatively small (otherwise there are too many possibilities to test for the correct combination of sum and product). With practice, you will get better at spotting when factoring is possible. However, if a quick check does not turn out a solution, then you can always fall back on the quadratic formula.

The next activity will give you some practice with the two methods for finding solutions for the quadratic equation.

Activity A4.1

Find the solutions to the quadratic equations below. Use either factorization or the quadratic formula.

a) $x^2 + 3x - 4 = 0$	$x_1 =$	$x_2 =$
b) $-3x^2 - 2x - \frac{3}{2} = 0$	$x_1 =$	$x_2 =$
c) $3x^2 + 5x = 0$	$x_1 =$	$x_2 =$
d) $-2x^2 - 4x + 4 = 0$	$x_1 =$	$x_2 =$
e) $x^2 + 4x - 4 = 0$	$x_1 =$	$x_2 =$
f) $2x^2 + 2x + \frac{1}{2} = 0$	$x_1 =$	$x_2 =$
g) $x^2 + 5x + 6 = 0$	$x_1 =$	$x_2 =$
h) $-x^2 - 3x + 10 = 0$	$x_1 =$	$x_2 =$