A2 Completing the Square

The expression $(x+d)^2$, a *binomial square*, can be expanded to $x^2+2dx+d^2$, a *perfect square trinomial*. In other words,

$$(x+d)^2 = x^2+2dx+d^2$$

On the other hand, if we are given an expression in the expanded form, then we can manipulate it to look like the condensed form. This method is called *completing the square*. The method is based on the fact that the constant *d* is half the multiplier of the variable *x* in the mixed term *2dx*. Therefore, we identify the multiplier and then divide by 2 to get the value of *d*. We will illustrate the method with the example from Section 3.2.

We start with the function in expanded form: $f(x)=\frac{1}{2}x^2-2x+5$. Since the equation above does not have a multiplicative factor for *x*, we need to first factor out any factor multiplied by x^2:

$$f(x)=\tfrac{1}{2}x^2-2x+5$$
$$=\tfrac{1}{2}\left(x^2-4x+10\right) \qquad \text{(since } -2=\tfrac{1}{2}\cdot(-4) \text{ and } 5=\tfrac{1}{2}\cdot 10\text{)}.$$

Since the constant *d* is half of the multiplicative factor of *x*, which is –2, we determine that *d* = -2. We can rewrite the expression inside the parenthesis so that it is the form x^2+2dx+something:

$$f(x)=\tfrac{1}{2}\left(x^2-2(-2)x+10\right)$$

Since *d* = -2, then $d^2=(-2)^2=4$. However, we have 10 as the additive constant. To fix this problem, we add and subtract d^2 to ensure that we have the proper constant that will complete the square. (We add <u>and</u> subtract so that we do not change the values of the expression; we are merely adding 0.) By putting it all together and simplifying, we have:

$$\begin{aligned}
f(x) &= \tfrac{1}{2}x^2-2x+5 \\
&= \tfrac{1}{2}[x^2-4+10] \\
&= \tfrac{1}{2}[x^2+2\cdot(-2)x+10] \\
&= \tfrac{1}{2}[x^2+2\cdot(-2)x+((-2)^2-(-2)^2)+10] \\
&= \tfrac{1}{2}[(x^2+2\cdot(-2)x+(-2)^2)+(-(-2)^2+10)] \\
&= \tfrac{1}{2}[(x-2)^2+(-4+10)] \\
&= \tfrac{1}{2}[(x-2)^2+6] \\
&= \tfrac{1}{2}(x-2)^2+3
\end{aligned}$$

Note that in the last step we have to multiply 6 by the factor ½ to get the final form. Forgetting this step is a common mistake.

Let's summarize:

Procedure for Completing the Square

1) The leading coefficient (the multiplier of the x^2 term) must be 1. If it is not, it needs to be factored out.
2) Determine the value of d by dividing the multiplier of x by 2. Write the mixed term in the form $2dx$.
3) Now, add and subtract d^2. (This will ensure that you do not change the function; essentially you are only adding 0.)
4) Group the terms of the perfect square trinomial and the remaining constants, respectively.
5) Write the perfect square trinomial in condensed form (as binomial square) and simplify the remaining constant terms.
6) If you factored out a multiplier in step 1), distribute that multiplier.
7) Check that your condensed form is equivalent to the original by expanding.

Below are several examples with step by step instructions, following the procedure above. After the final step, the answer is checked by expanding the result.

Example 1:

$f(x)$	$= x^2 + 8x + 12$	Original function
	$= x^2 + 2 \cdot 4x + 12$	Step 2; $d = 4$
	$= x^2 + 2 \cdot 4x + (4^2 - 4^2) + 12$	Step 3
	$= (x^2 + 2 \cdot 4x + 4^2) + (-4^2 + 12)$	Step 4
	$= (x+4)^2 + (-16 + 12)$	Step 5
	$= (x+4)^2 - 4$	Simplify

Check: $(x+4)^2 - 4 = (x^2 + 8x + 16) - 4 = x^2 + 8x + 12$

Example 2:

$f(x)$	$= 2x^2 - 4x - 8$	Original function
	$= 2[x^2 - 2x - 4]$	Step 1
	$= 2[x^2 + 2\cdot(-1)x - 4]$	Step 2; $d = -1$
	$= 2[x^2 + 2\cdot(-1)x + ((-1)^2 - (-1)^2) - 4]$	Step 3
	$= 2[(x^2 + 2\cdot(-1)^2 + (-1)^2) + (-(-1)^2 - 4)]$	Step 4
	$= 2[(x-1)^2 - 5]$	Step 5
	$= 2(x-1)^2 - 10$	Step6

Check: $2(x-1)^2 - 10 = 2(x^2 - 2x + 1) - 10 = 2x^2 - 4x + 2 - 10 = 2x^2 - 4x - 8$

Example 3:

$f(x)$	$= -x^2 - 3x + 1$	Original function
	$= -[x^2 + 3x - 1]$	Step 1
	$= -[x^2 + 2\cdot\frac{3}{2}x - 1]$	Step 2
	$= -[x^2 + 2\cdot\frac{3}{2}x + ((\frac{3}{2})^2 - (\frac{3}{2})^2) - 1]$	Step 3
	$= -[(x^2 + 2\cdot\frac{3}{2}x + (\frac{3}{2})^2) + (-(\frac{3}{2})^2 - 1)]$	Step 4
	$= -[(x + \frac{3}{2})^2 + (-\frac{9}{4} - \frac{4}{4})]$	Step 5 and find LCD
	$= -[(x + \frac{3}{2})^2 - \frac{13}{4}]$	Simplify
	$= -(x + \frac{3}{2})^2 + \frac{13}{4}$	Step 6

Check: $-(x + \frac{3}{2})^2 + \frac{13}{4} = -(x^2 + 2\frac{3}{2}x + \frac{9}{4}) + \frac{13}{4} = -x^2 - 3x - \frac{9}{4} + \frac{13}{4} = -x^2 - 3x + \frac{4}{4} = -x^2 - 3x + 1$

There is a geometrical interpretation associated with the completion of the square. We begin with a figure that consists of a square of length x and two rectangles with sides of length x and d. The square has an area of x^2 and each rectangle has an area of $d \cdot x$. Thus, we have a total area of $x^2 + 2 \cdot d \cdot x$. When we add the small black square (with sides of length d and area d^2) we've completed the (large) square with sides (x+d) and area $(x+d)^2$!